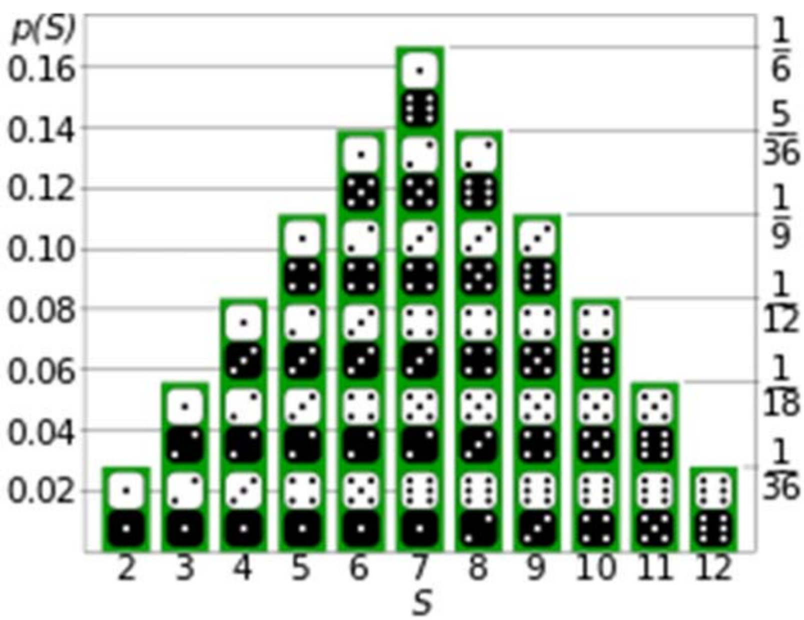
Discrete Probability Distribution:

If a variable x can assume a discrete set of values $x_1, x_2, \cdots, x_k$ with respective probabilities $p_1, p_2, \cdots, p_k$ where $p_1 + p_2 + \cdots + p_k = 1$, we say that a discrete probability distribution for x has been defined. The function (x) , which has the respective values $p_1, p_2, \cdots, p_k$ for $x_1, x_2, \cdots, x_k$ is called the probability function, or frequency function, of x. Because x can assume certain values with given probabilities, it is often called a discrete random variable. A random variable is also known as a chance variable or stochastic variable.

Example: Let a pair of fair dice be tossed and let x denote the sum of the points obtained. Then the probability distribution is as shown in below:

x	2	3	4	5	6	7	8	9	10	11	12
$P(x)$	$^1/_{36}$	$^2/_{36}$	$^3/_{36}$	$^4/_{36}$	$^5/_{36}$	$^6/_{36}$	$^5/_{36}$	$^4/_{36}$	$^3/_{36}$	$^2/_{36}$	$^1/_{36}$



Binomial Distribution:

Given a choice of fruits, apple (A) or banana (B), let P(A) = p and P(B) = q.

In choosing one fruit, the sample space and corresponding probabilities are: $\begin{matrix}\{A, B\} \\ \{p, q\}\end{matrix}$

In the case of one trial, the variable is a Bernoulli random variable. With two fruits (and AB = BA): $\begin{Bmatrix} AA & AB & BB \\ p^2 & 2pq & q^2 \end{Bmatrix}$ The coefficients in the probabilities are equal to the number of ways that the outcome can be obtained.

$\text{i.e.}, \{\text{Success} = event = p\}\ and\ \{\text{Failure} = no\ event = q\}$

Binomial expansion summarizes this result: $(p+q)^n$ where n is the sample size. The probability mass function of a binomial distribution is:

$$P(X=x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x(n-x)!} p^x (1-p)^{n-x}$$

Where x is the number of "successes" (here, the number of apples). The binomial distribution is the expected distribution of outcomes in random samples of size n, with probability p of success. Mean and variance of binomial distribution: $= np$, $\sigma = \sqrt{npq}$ and $\sigma^2 = npq$.

Example: Suppose a biased coin comes up heads with probability 0.3 when tossed. What is the probability of achieving $\{0, 1, \ldots, 6\}$ heads after six tosses?

Solve

$$Pr(0\ heads) = f(0) = Pr(x=0) = \binom{6}{0} 0.3^0 (1-0.3)^{6-0} = 0.117649$$

$$Pr(1\ heads) = f(1) = Pr(x=1) = \binom{6}{1} 0.3^1 (1-0.3)^{6-1} = 0.302526$$

$$Pr(2\ heads) = f(2) = Pr(x=2) = \binom{6}{2} 0.3^2 (1-0.3)^{6-2} = 0.324135$$

$$Pr(3\ heads) = f(3) = Pr(x=3) = \binom{6}{3} 0.3^3 (1-0.3)^{6-3} = 0.185220$$

$$Pr(4\ heads) = f(4) = Pr(x=4) = \binom{6}{4} 0.3^4 (1-0.3)^{6-4} = 0.059535$$

$$Pr(5\ heads) = f(5) = Pr(x=5) = \binom{6}{5} 0.3^5 (1-0.3)^{6-5} = 0.010206$$

$$Pr(6\ heads) = f(6) = Pr(x=6) = \binom{6}{6} 0.3^6 (1-0.3)^{6-6} = 0.000729$$

Poisson Distribution:

The Poisson distribution can be used to approximate the Binomial distribution when one event is rare($p < 0.1$), and the sample size is large ($np > 5$). A Poisson variable Y must be:

1. Rare: Small mean relative to the number of possible events per sample;
2. Random: Independent of previous occurrences in the sample.

This distribution can model the number of times that a rare event occurs, and test whether rare events are independent of each other.

The parameter λ is the expected number of successes. If X is binomial with large n and small p, the number of success is approximately a Poisson random variable with λ= np. The probability mass function of a Poisson distribution is:

$$P(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}$$

λ is the only parameter needed to describe a Poisson distribution. It is equal to both the variance and the mean: $\lambda = \mu = \sigma^2$.

The Birthday Paradox Distribution:

If there are n people in a room, there are $\binom{n}{2}$ pairs of people. We define a success as having one pair share a birthday, with probability 1/365. Thus $n = \binom{n}{2}$ $p = 1/365$ The expected number of successes is: $\mu = np = \binom{n}{2}/365 = \frac{n(n-1)}{730} = \lambda$

Thus the probability that no two people share a birthday is:

$$P(X = 0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-\lambda} = exp\left\{\frac{-n(n-1)}{730}\right\}$$

If we want to find the number of people for which the probability is less than 0.5:

$$exp\left\{\frac{-n(n-1)}{730}\right\} \leq \frac{1}{2}$$

$$exp\left\{\frac{-n(n-1)}{730}\right\} \geq 2$$

$$n(n-1) \geq 730\ \ln(2)$$

Which is solved with $n \approx 23$, meaning that if there are 23 people in a room, there is a probability of 0.5 that two of them share a birthday.