AL-Karkh University of Science
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Department of Geophysics

Mathematics

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References:

1- Mathematical Methods Dr. T.K.V. Iyengar , Dr. B. Krishna Gandhi 2008

Chapter 10

Fourier Series

10.1 INTRODUCTION

Suppose that a given function $f(x)$ defined in $[-\pi, \pi]$ or $[0, 2\pi]$ or in any other interval can be expressed as a trigonometric series as

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots + a_n \cos nx + \ldots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots + b_n \sin nx + \ldots$$

$$= \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + (a_3 \cos 3x + b_3 \sin 3x) + \ldots +$$
$$(a_n \cos nx + b_n \sin nx) + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(A)$$

where the a's and b's are constants within a desired range of values of the variable. Such series is known as the **Fourier series** for $f(x)$ and the constants a_0, a_n, b_n $(n = 1, 2, 3 \ldots)$ are called Fourier coefficients of $f(x)$.

Every term of (A) except the first, has period 2π and hence any function represented by a series of the above form will also be periodic with period 2π.

10.2 PERIODIC FUNCTIONS

[JNTU 2003, 2003S]

A function $f(x)$ is said to be of period T or to be periodic with period $T > 0$ if for all x, $f(x + T) = f(x)$, and T is the least of such values.

Ex. 1: Since $\sin x = \sin(x + 2\pi) = \sin(x + 4\pi) = \sin(x + 6\pi) = \ldots,$ the function $\sin x$ is periodic with period 2π. There is no positive value T, $0 < T < 2\pi$ such that $\sin(x + T) = \sin x$ for all x.

In a similar manner the period of $\cos x$ is 2π.

Ex. 2: The period of $\tan x$ is π.

Ex. 3: $\sin 3x = \sin(2\pi + 3x) = \sin 3\left(\frac{2\pi}{3} + x\right)$

We notice that $\sin 3x$ is periodic with period $\frac{2\pi}{3}$.

10.3 EULER'S FORMULAE

The Fourier Series for the function $f(x)$ in the interval $C \le x \le C + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\left.\begin{aligned} \text{where } a_0 &= \frac{1}{\pi}\int_C^{C+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\,dx \\ \text{and} \quad b_n &= \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx \end{aligned}\right\} \quad ...(A)$$

These values of a_0, a_n, b_n are known as Euler's formulae.

Proof. Let $f(x)$ be represented in the interval $[C, C+2\pi]$ by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad ...(1)$$

Let us assume that the series is uniformly convergent in the interval $C \le x \le C + 2\pi$. Then the series can be integrated term by term.

To evaluate a_0 :

Integrating both sides of (1) from $x = C$ to $x = C + 2\pi$, we get

$$\int_C^{C+2\pi} f(x)\,dx = \frac{a_0}{2}\int_C^{C+2\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_C^{C+2\pi} \cos nx\,dx + b_n \int_C^{C+2\pi} \sin nx\,dx\right]$$

$$= \frac{a_0}{2}(C + 2\pi - C) + \sum_{n=1}^{\infty}\left[a_n\left(\frac{\sin nx}{n}\right)_C^{C+2\pi} + b_n\left(\frac{-\cos nx}{n}\right)_C^{C+2\pi}\right].$$

$$= a_0\pi + \sum_{n=1}^{\infty}\left[a_n.0 - b_n.0\right]$$

$$= a_0\pi$$

Hence $$\boxed{a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\,dx} \quad ...(2)$$

To evaluate a_n :

Multiplying both sides of (1) by $\cos mx$ and then integrating from $x = C$ to $x = C + 2\pi$, we get

$$\int_C^{C+2\pi} f(x)\cos mx\,dx = \frac{a_0}{2}\int_C^{C+2\pi} \cos mx\,dx + \sum_{n=1}^{\infty} a_n\left[\int_C^{C+2\pi} \cos nx \cos mx\,dx\right] + \sum_{n=1}^{\infty} b_n\left[\int_C^{C+2\pi} \sin nx \cos mx\,dx\right]$$

The first and third integrals on the right-hand side are always zero, but second integral is equal to π when $m = n$; otherwise it also vanishes when $m \neq n$. Hence

$$\int_C^{C+2\pi} f(x)\cos nx\,dx = \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n\int_C^{C+2\pi} 2\cos^2 nx\,dx\right] = \frac{a_n}{2}\int_C^{C+2\pi}(1 + \cos 2nx)\,dx$$

$$= \frac{a_n}{2}\left(x + \frac{\sin 2nx}{2n}\right)_c^{c+2\pi} = \frac{a_n}{2}(2\pi) = a_n\pi$$

$$\boxed{\therefore\ a_n = \frac{1}{\pi}\int_{C}^{C+2\pi} f(x)\cos nx\, dx} \qquad (\because\ m = n) \qquad \text{...(3}$$

To evaluate b_n :

Multiplying both sides of (1) by $\sin mx$ and then integrating from $x = C$ to $x = C + 2\pi$ we get

$$\int_{C}^{C+2\pi} f(x)\sin mx\, dx = \frac{a_0}{2}\int_{C}^{C+2\pi}\sin mx\, dx + \sum_{n=1}^{\infty}\left[a_n\int_{C}^{C+2\pi}\cos nx\sin mx\, dx + b_n\int_{C}^{C+2\pi}\sin nx\sin mx\, dx\right]$$

The first two integrals on the right hand side are always zero, but the third integral i equal to π when $m = n$; otherwise it also vanishes when $m \neq n$. Hence

$$\int_{C}^{C+2\pi} f(x)\sin nx\, dx = \frac{a_0}{2}(0) + a_n(0) + \frac{1}{2}b_n\int_{C}^{C+2\pi} 2\sin^2 nx\, dx$$

$$= \frac{1}{2}b_n\int_{C}^{C+2\pi}(1-\cos 2nx)\, dx = \frac{1}{2}b_n\left(x - \frac{\sin 2nx}{2n}\right)_{C}^{C+2\pi}$$

$$= \frac{1}{2}b_n(2\pi) = \pi b_n$$

$$\boxed{\therefore\ b_n = \frac{1}{\pi}\int_{C}^{C+2\pi} f(x)\sin nx\, dx} \qquad \text{...(4)}$$

The results (2), (3) and (4) are called Euler's formulae.

Cor. 1 : If $f(x)$ is to be expanded as a Fourier series in the interval $0 \leq x \leq 2\pi$, p $C = 0$, then the formulae (A) reduces to

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{0}^{2\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos nx\, dx \\ b_n &= \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin nx\, dx \end{aligned}\right\} \qquad \text{...(B)}$$

Cor. 2 : If $f(x)$ is to be expanded as a Fourier series in $[-\pi, \pi]$, p $C = -\pi$; the interval becomes $-\pi \leq x \leq \pi$ and the formulae (A) reduces to

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx \end{aligned}\right\} \qquad \text{...(C}$$

In many of the problems, we come across expansions of $f(x)$ in $[0, 2\pi]$ or $[-\pi, \pi]$ and hence formulae [B] and [C] to be remembered carefully.

10.4 CONDITIONS FOR FOURIER EXPANSION [JNTU 2003,2004]

Dirichlet has formulated certain conditions known as Dirichlet conditions under which certain functions possess valid Fourier expansions.

A function $f(x)$ has a valid Fourier series expansion of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where a_0, a_n, b_n are constants, provided:

(*i*) $f(x)$ is well defined and single-valued, except possibly at a finite number of points in the interval of definition.

(*ii*) $f(x)$ has a finite number of finite discontinuities in the interval of definition.

(*iii*) $f(x)$ has at most a finite number of maxima and minima in the interval of definition.

Note : The above conditions are sufficient but not necessary.

10.5 FUNCTIONS HAVING POINTS OF DISCONTINUITY

In deriving the Euler's formulae for a_0, a_n, b_n it was assumed that $f(x)$ is continuous. Instead a function may have a finite number of discontinuities. Even then such a function is expressible as a Fourier series.

For instance, let the function $f(x)$ be defined by

$$f(x) = \phi(x),\ C < x < x_0$$
$$= \Psi(x),\ x_0 < x < C + 2\pi$$

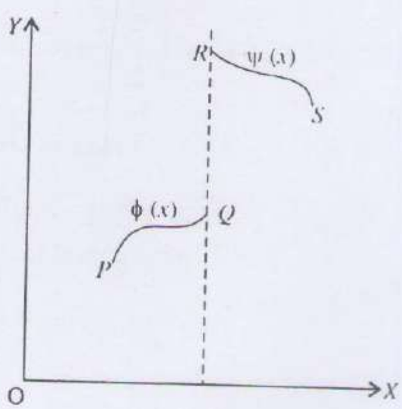
where x_0 is the point of discontinuity in $(C, C + 2\pi)$.

In such cases also we obtain the Fourier series for $f(x)$ in the usual way. The values of a_0, a_n, b_n are given by

$$a_0 = \frac{1}{\pi}\left[\int_C^{x_0}\phi(x)\,dx + \int_{x_0}^{C+2\pi}\Psi(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_C^{x_0}\phi(x)\cos nx\,dx + \int_{x_0}^{C+2\pi}\Psi(x)\cos nx\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_C^{x_0}\phi(x)\sin nx\,dx + \int_{x_0}^{C+2\pi}\Psi(x)\sin nx\,dx\right]$$



It may be seen from the graph, that at a point of finite discontinuity $x = x_0$, there is a finite jump (= QR) in the value of the function $f(x)$ at $x = x_0$. Both the limits on the right and left exist and are different. At such a point, the Fourier series converges to

$$\frac{1}{2}\left[f(x_0 - 0) + f(x_0 + 0)\right]$$

In fact, if $f(x)$ satisfies Dirichlet's conditions and

$f(x) \simeq \frac{1}{2}a_0 + \Sigma(a_n \cos nx + b_n \sin nx)$ in $[C, C+2\pi]$ then the right-hand side series converges to $f(x)$ if x is a point of continuity of $f(x)$ and converges to $\frac{1}{2}[f(x+0)+f(x-0)]$ if x is a point of discontinuity of $f(x)$.

This result is useful to determine the sums of certain series.

Note 1. (*i*) $\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & \text{for } m \neq n \\ \pi, & \text{for } m = n > 0 \\ 2\pi, & \text{for } m = n = 0 \end{cases}$

(*ii*) $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & \text{for } m \neq n \text{ and } m = n = 0 \\ \pi, & \text{for } m = n > 0 \end{cases}$

Note 2. (*i*) $\sin n\pi = 0$, $\sin 2n\pi = 0$ *for* $n \in Z$, where Z is the set of all integers

(*ii*) $\cos n\pi = (-1)^n$, $\cos 2n\pi = 1$, $n \in Z$

(*iii*) $\sin\left(n + \frac{1}{2}\right)\pi = (-1)^n, n \in Z$

(*iv*) $\cos\left(n + \frac{1}{2}\right)\pi = 0, n \in Z$

EXAMPLES

Ex.1: Express $f(x) = x-\pi$ as Fourier series in the interval $-\pi < x < \pi$.

Sol. Let the function $x-\pi$ be represented by the Fourier series

$$x - \pi = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \dots (1)$$

Then $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \, dx - \pi\int_{-\pi}^{\pi} dx\right]$$

$$= \frac{1}{\pi}\left[0 - \pi \cdot 2\int_0^{\pi} dx\right] \quad (\because x \text{ is odd function})$$

$$= \frac{1}{\pi}\left[-2\pi(x)_0^{\pi}\right] = -2(\pi - 0) = -2\pi$$

and $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\cos nx \, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x \cos nx \, dx - \pi\int_{-\pi}^{\pi} \cos nx \, dx\right]$$

$$= \frac{1}{\pi}\left[0 - 2\pi\int_0^\pi \cos nx\, dx\right]$$

[$\because$ $x \cos nx$ is odd function and $\cos nx$ is even function]

$$\therefore\ a_n = -2\int_0^\pi \cos nx\, dx = -2\left(\frac{\sin nx}{n}\right)_0^\pi$$

$$= -\frac{2}{n}(\sin n\pi - \sin 0) = -\frac{2}{n}(0-0)$$

$$= 0 \text{ for } n = 1, 2, 3, \ldots.$$

Finally $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-\pi)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\sin nx\, dx - \pi\int_{-\pi}^{\pi}\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[2\int_0^\pi x\sin nx\, dx - \pi(0)\right]$$

[$\because$ $x \sin nx$ is even function and $\sin nx$ is odd function]

$$\therefore\ b_n = \frac{2}{\pi}\int_0^\pi x\sin nx\, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\left(\frac{-\pi\cos n\pi}{n} + 0\right) - (0+0)\right] \quad (\because\ \sin n\pi = 0)$$

$$= \frac{-2}{n}\cos n\pi = \frac{-2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} \text{ for } n = 1, 2, 3, \ldots$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$x - \pi = -\pi + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin nx$$

$$= -\pi + 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right]$$

This is the required Fourier series.

Ex. 2: Find a Fourier series to represent $f(x) = x^2$ in the interval $(0, 2\pi)$.

Sol. Let $$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots (1)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x^2 dx$$

$$= \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{2\pi} = \frac{1}{3\pi}(8\pi^3 - 0) = \frac{8}{3}\pi^2$$

and $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(0 + \frac{4\pi\cos 2n\pi}{n^2} - 0\right) - (0+0-0)\right]$$

$$= \frac{4}{n^2} \quad (\because \cos 2n\pi = 1)$$

Finally $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{4\pi^2\cos 2n\pi}{n} + 0 + \frac{2\cos 2n\pi}{n^3}\right) - \left(0 + 0 + \frac{2}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right] = \frac{-4\pi}{n}$$

Substituting the values of a_0, a_n and b_n in (1) we get

$$x^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{4\pi}{n}\sin nx$$

$$= \frac{4}{3}\pi^2 + 4\left(\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right) - 4\pi\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right)$$

Ex.3. Determine the Fourier series expansion of the function $f(x) = \frac{1}{12}(3x^2 - 6x\pi + 2\pi^2)$ in the interval $(0, 2\pi)$.

Sol. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{12}(3x^2 - 6x\pi + 2\pi^2)dx$$

$$= \frac{1}{12\pi}\left(x^3 - 3x^2\pi + 2\pi^2 x\right)_0^{2\pi}$$

$$= \frac{1}{12\pi}(8\pi^3 - 12\pi^3 + 4\pi^3) = 0$$

and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{12}(3x^2-6x\pi+2\pi^2).\cos nx\,dx$

$$= \frac{1}{12\pi}\left[(3x^2-6x\pi+2\pi^2)\left(\frac{\sin nx}{n}\right)-(6x-6\pi)\left(\frac{-\cos nx}{n^2}\right)+6\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[\frac{12\pi-6\pi}{n^2}+\frac{6\pi}{n^2}\right]=\frac{1}{n^2}$$

Finally $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

$$= \frac{1}{12\pi}\int_0^{2\pi}(3x^2-6x\pi+2\pi^2)\cdot\sin nx\,dx$$

$$= \frac{1}{12\pi}\left[(3x^2-6x\pi+2\pi^2)\cdot\left(\frac{-\cos nx}{n}\right)-(6x-6\pi)\left(\frac{-\sin nx}{n^2}\right)+6\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= 0$$

Hence $f(x) = \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$

i.e. $\frac{1}{12}(3x^2-6x\pi+2\pi^2) = \frac{1}{1^2}\cos x+\frac{1}{2^2}\cos 2x+\frac{1}{3^2}\cos 3x+\ldots$

Ex. 4: Obtain the Fourier series for the function $f(x) = e^x$ from $x = 0$ to $x = 2\pi$.

Sol. Let $e^x = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ ...(1)

Then $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} e^x dx$

$$= \frac{1}{\pi}\left(e^x\right)_0^{2\pi} = \frac{1}{\pi}(e^{2\pi}-1)$$

and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x\cos nx\,dx$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx+n\sin nx)\right]_0^{2\pi}$$

$$= \frac{e^{2\pi}-1}{\pi(1+n^2)}\quad (\because \sin 2n\pi = 0,\ \cos 2n\pi = 1)$$

Finally $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x\sin nx\,dx$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx-n\cos nx)\right]_0^{2\pi} = \frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}$$

Hence $$e^x = \frac{e^{2\pi}-1}{\pi} + \sum_{n=1}^{\infty} \frac{e^{2\pi}-1}{\pi(1+n^2)}\cos nx + \sum_{n=1}^{\infty} \frac{(-n)(e^{2\pi}-1)}{\pi(1+n^2)}\sin nx$$

$$= \frac{e^{2\pi}-1}{2\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx}{1+n^2} - \sum_{n=1}^{\infty}\frac{n\sin nx}{1+n^2}\right]$$

This is the required Fourier series.

Ex. 5: Find the Fourier series representing $f(x) = x$, $0 < x < 2\pi$. Sketch the graph of $f(x)$ from -4π to 4π. **[JNTU 2003 (Set No. 4), 2005 (Set No.1)]**

Sol. Let $$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x\, dx = \frac{1}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi} = 2\pi$$

and $$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\, dx = \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right)_0^{2\pi}$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2}\cos 2n\pi - \frac{1}{n^2}\right) = \frac{1}{\pi}\left(\frac{1}{n^2} - \frac{1}{n^2}\right) = 0 \quad [\because \cos 2n\pi = \cos 0 = 1]$$

Finally $$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin nx\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{-1}{n}x\cos nx + \frac{1}{n^2}\sin nx\right]_0^{2\pi}$$

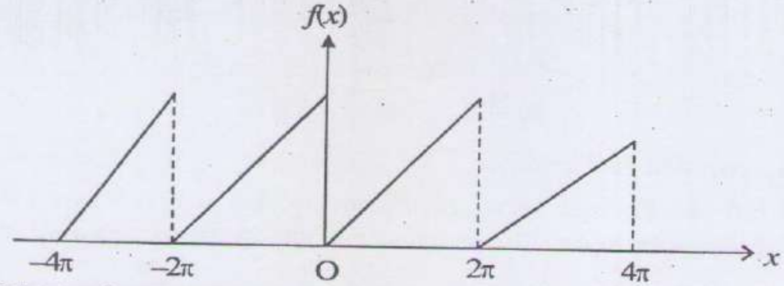
$$= \frac{1}{\pi}\left[\left(\frac{-1}{n}2\pi\cos 2n\pi + 0\right) - (0+0)\right] = \frac{-2}{n} \quad (\because \cos 2n\pi = 1)$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$x = \pi - 2\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$= \pi - 2\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right) \qquad \ldots(2)$$

Graph of $f(x) = x$ in $[-4\pi, 4\pi]$



Ex. 6: Obtain the Fourier series for $f(x) = x - x^2$ in the interval $[-\pi, \pi]$. Hence show that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

(or) $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

[JNTU Dec. 2002 (Set No. 1)]

Sol. The Fourier series of $f(x) = x - x^2$ in $[-\pi, \pi]$ is given by

$$x - x^2 \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(1)$$

Using Euler's formulae, we determine a_n and b_n.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} x\,dx - \int_{-\pi}^{\pi} x^2 dx\right]$$

$$= \frac{1}{\pi}\left[0 - 2\int_0^{\pi} x^2 dx\right] \quad (\because x \text{ is odd function and } x^2 \text{ is even function})$$

$$= \frac{-2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{-2}{3\pi}(\pi^3 - 0) = -\frac{2\pi^2}{3}$$

and $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\frac{\sin nx}{n} - (1 - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi} \text{ by parts}$$

$$= \frac{-4\cos n\pi}{n^2} = \frac{-4(-1)^n}{n^2} \; (n \neq 0) \quad [\because \cos n\pi = (-1)^n]$$

Finally $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x - x^2)\left(\frac{-\cos nx}{n}\right) - (1 - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-2\cos n\pi}{n} = \frac{-2(-1)^n}{n}$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$x - x^2 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{-4(-1)^n}{n^2}\cos nx + \frac{-2(-1)^n \sin nx}{n}\right)$$

$$= \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[4\frac{(-1)^{n+1}}{n^2}\cos nx + 2\frac{(-1)^{n+1}}{n}\sin nx\right]$$

$$= \frac{-\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\right]$$

$$+2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right] \quad \ldots(2)$$

Deduction : $x = 0$ is a point of continuity of $f(x)$. Hence the Fourier series of $f(x)$ at $x = 0$ converges to $f(0)$.

Putting $x = 0$ in (2), we get

$$0 = \frac{-\pi^2}{3} + 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right) \text{ i.e., } \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$$

or $$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

Ex. 7: Expand $f(x) = \left(\frac{\pi - x}{2}\right)^2$, $0 < x < 2\pi$ in a Fourier series.

[JNTU 2004 (Set No. 4)]

Sol. Given $f(x) = \left(\frac{\pi - x}{2}\right)^2 = \frac{(\pi - x)^2}{4}$ *in* $(0, 2\pi)$.

The Fourier series of $f(x)$ in $[0, 2\pi]$ is given by

$$\frac{(\pi - x)^2}{4} \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \quad \ldots(1)$$

where a_n, b_n are obtained through Euler's formulae.

Then $$a_0 = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi - x)^2}{4}\,dx = \frac{1}{4\pi}\left[-\frac{1}{3}(\pi - x)^3\right]_0^{2\pi}$$

$$= \frac{-1}{12\pi}\left[(-\pi^3) - \pi^3\right] = \frac{\pi^2}{6}$$

and $$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi - x)^2}{4}\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}, \text{ by parts}$$

$$= \frac{1}{4\pi}\left[\left(0 + \frac{2\pi\cos 2n\pi}{n^2} + 0\right) - \left(0 - \frac{2\pi\cos 0}{n^2} + 0\right)\right]$$

$$= \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] \quad [\because \cos 2n\pi = \cos 0 = 1]$$

$$= \frac{1}{4\pi}\left(\frac{4\pi}{n^2}\right) = \frac{1}{n^2} \quad (\text{if } n \neq 0).$$

Finally $b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\sin nx\,dx$

$$= \frac{1}{4\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right)-2(\pi-x)(-1)\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2\cos 2n\pi}{n}-0+2\frac{\cos 2n\pi}{n^3}\right)-\left(\frac{-\pi^2}{n}-0+\frac{2}{n^3}\right)\right]$$

$$= \frac{1}{4\pi}\left[\left(\frac{-\pi^2}{n}+\frac{2}{n^3}\right)-\left(\frac{-\pi^2}{n}+\frac{2}{n^3}\right)\right] \quad (\because \cos 2n\pi = 1)$$

$$= 0$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$\left(\frac{\pi-x}{2}\right)^2 = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx = \frac{\pi^2}{12}+\frac{\cos x}{1^2}+\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}+\ldots$$

which is the required Fourier series.

Ex. 8: Find the Fourier series to represent the function e^{-ax} from $x = -\pi$ to $x = \pi$. Deduce from this that

$$\frac{\pi}{\sinh \pi} = 2\left[\frac{1}{2^2+1}-\frac{1}{3^2+1}+\frac{1}{4^2+1}-\ldots\right]$$

Sol. Let the function e^{-ax} be represented by the Fourier series

$$e^{-ax} = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx) \qquad \ldots(1)$$

Then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}e^{-ax}\,dx = \frac{1}{\pi}\left(\frac{e^{-ax}}{-a}\right)_{-\pi}^{\pi} = \frac{-1}{a\pi}\left(e^{-a\pi}-e^{a\pi}\right) = \frac{e^{a\pi}-e^{-a\pi}}{a\pi}$

$$\therefore \frac{a_0}{2} = \frac{1}{a\pi}\left(\frac{e^{a\pi}-e^{-a\pi}}{2}\right) = \frac{\sinh a\pi}{a\pi}$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}e^{-ax}\cos nx\,dx$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\cos nx+n\sin nx)\right]_{-\pi}^{\pi}$$

[using $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$]

$$\therefore \quad a_n = \frac{1}{\pi}\left\{\frac{e^{-a\pi}}{a^2+n^2}(-a\cos n\pi+0)-\frac{e^{a\pi}}{a^2+n^2}(-a\cos n\pi+0)\right\} \qquad [\because \sin n\pi = 0]$$

$$= \frac{a}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi})\cos n\pi = \frac{2a\cos n\pi \sinh a\pi}{\pi(a^2+n^2)}$$

$$= \frac{(-1)^n 2a \sinh a\pi}{\pi(a^2+n^2)} \quad [\because \cos n\pi = (-1)^n]$$

Finally $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\, dx$

$$\left[\text{Use the formula } \int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$\therefore \quad b_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{-a\pi}}{a^2+n^2}(0 - n\cos n\pi) - \frac{e^{a\pi}}{a^2+n^2}(0 - n\cos n\pi)\right]$$

$$= \frac{n\cos n\pi\ (e^{a\pi} - e^{-a\pi})}{\pi(a^2+n^2)} = \frac{(-1)^n 2n\sinh a\pi}{\pi(a^2+n^2)}$$

Substituting the values of $\frac{a_0}{2}$, a_n and b_n in (1), we get

$$e^{-ax} = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n 2a\sinh a\pi}{\pi(a^2+n^2)}\cos nx + \frac{(-1)^n 2n\sinh a\pi}{\pi(a^2+n^2)}\sin nx\right]$$

$$= \frac{2\sinh a\pi}{\pi}\left\{\left(\frac{1}{2a} - \frac{a\cos x}{1^2+a^2} + \frac{a\cos 2x}{2^2+a^2} - \frac{a\cos 3x}{3^2+a^2} + \dots\right)\right.$$

$$\left. - \left(\frac{\sin x}{1^2+a^2} - \frac{2\sin 2x}{2^2+a^2} + \frac{3\sin 3x}{3^2+a^2} - \dots\right)\right\} \quad \dots(2)$$

which is the required Fourier series.

Deduction

Putting $x = 0$ and $a = 1$ in (2), we get

$$1 = \frac{2\sinh\pi}{\pi}\left\{\frac{1}{2} - \frac{1}{2} + \frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \dots\right\}$$

or $$\frac{\pi}{\sinh\pi} = 2\left(\frac{1}{2^2+1} - \frac{1}{3^2+1} + \frac{1}{4^2+1} - \dots\right)$$

which is the required result.

(Notice that $x = 0$ is a point of continuity of $f(x) = e^{-ax}$).

Ex. 9: Expand $f(x) = x\sin x$, $0 < x < 2\pi$ as a Fourier Series

[JNTU 2004, 2006 (Set No.3)]

Sol. Let $x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \quad \dots(1)$

Then $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \, dx$

$$= \frac{1}{\pi}\left[x(-\cos x) - 1\cdot(-\sin x)\right]_0^{2\pi} = \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{2\pi}$$

$$= \frac{1}{\pi}[(-2\pi + 0) - (0+0)] = -2$$

and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x \cos nx \, dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\,(2\sin x\cos nx)dx = \frac{1}{2\pi}\int_0^{2\pi} x\,[\sin(n+1)x - \sin(n-1)x]\,dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\cdot\left\{\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi} \quad (n \neq 1)$$

$$= \frac{1}{2\pi}\left[2\pi\left\{-\frac{\cos 2(n+1)\pi}{n+1} + \frac{\cos 2(n-1)\pi}{n-1}\right\}\right] \quad (n \neq 1)$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2-1} \quad (n \neq 1)$$

If $n = 1$, we have

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x \, dx = \frac{1}{2\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\cdot\left(\frac{-\sin 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}(-\pi) = -\frac{1}{2}$$

Finally $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{2\pi} x\sin x \sin nx \, dx$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x \sin nx)dx \qquad \ldots (2)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(n-1)x - \cos(n+1)x]dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right\} - 1\cdot\left\{\frac{-\cos(n-1)x}{(n-1)^2} + \frac{\cos(n+1)x}{(n+1)^2}\right\}\right]_0^{2\pi} \quad (n \neq 1)$$

$$= \frac{1}{2\pi}\left[\frac{\cos 2(n-1)\pi}{(n-1)^2} - \frac{\cos 2(n+1)\pi}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] \quad (n \neq 1)$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right] \quad (n \neq 1)$$

$\therefore \; b_n = 0 \;$ for $\; n \neq 1$

If $n = 1$, then

$$b_1 = \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin^2 x \, dx \qquad \text{[Putting } n = 1 \text{ in (2)]}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1\cdot\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi \cdot 2\pi - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \pi$$

Substituting the values of a_0, a_n, and b_n in (1), we get

$$x \sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2-1}\cos nx + \pi \sin x$$

$$= -1 + \pi \sin x - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{\cos nx}{n^2-1}$$

This is the required Fourier series.

Ex. 10: Find the Fourier series to represent the function $f(x)$ given by

$$f(x) = -k, \text{ for } -\pi < x < 0$$
$$= k, \text{ for } 0 < x < \pi$$

Hence show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$ **[JNTU Dec. 2002, 2005 (Set No. 4)]**

Sol. The required series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad \ldots(1)$$

Then $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\,dx + \int_0^{\pi} k\,dx\right]$

$$= \frac{k}{\pi}\left[-(x)_{-\pi}^{0} + (x)_0^{\pi}\right] = \frac{k}{\pi}(-\pi + \pi) = 0$$

and $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_0^{\pi} k\cos nx\,dx\right]$$

$$= \frac{k}{\pi}\left[-\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left(\frac{\sin nx}{n}\right)_0^{\pi}\right] = \frac{k}{\pi}(0+0) = 0$$

Finally $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -k\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right] = \frac{k}{\pi}\left[\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} - \left(\frac{\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{k}{\pi}\left[\left(\frac{1-(-1)^n}{n}\right) - \left(\frac{(-1)^n - 1}{n}\right)\right] \quad [\because \cos n\pi = (-1)^n]$$

$$= \frac{k}{\pi}\left(\frac{2-2(-1)^n}{n}\right) = \frac{4k}{n\pi}, \text{ when } n \text{ is odd.}$$

Substituting the values of a_0, a_n, b_n in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{4k}{n\pi}\sin nx, \quad \text{where } n \text{ is odd}$$

$$= \frac{4k}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right] \qquad \ldots(2)$$

Deduction : At $x = \frac{\pi}{2}$, $f(x)$ is continuous. Hence the Fourier series converges to $f\left(\frac{\pi}{2}\right)$. Thus putting $x = \frac{\pi}{2}$ in (2), we get

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \ldots\right)$$

i.e., $1 = \frac{4}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right)$ *i.e.,* $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

Ex. 11: Find the Fourier series of the following function.

$$f(x) = 0, \; -\pi \le x \le 0$$
$$= \frac{\pi x}{4}, \; 0 < x < \pi$$

Also deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Sol. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \qquad \ldots(1)$$

Then $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\cdot dx + \int_{0}^{\pi}\frac{\pi x}{4}dx\right]$

$$= \frac{1}{\pi}\cdot\frac{\pi}{4}\int_{0}^{\pi} x\,dx = \frac{1}{4}\left(\frac{x^2}{2}\right)_0^{\pi} = \frac{\pi^2}{8}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot \cos nx \, dx + \int_{0}^{\pi} \frac{\pi x}{4} \cos nx \, dx\right] = \frac{1}{\pi} \cdot \frac{\pi}{4} \int_{0}^{\pi} x \cos nx \, dx$$

$$= \frac{1}{4}\left[x\left(\frac{\sin nx}{n}\right) - 1.\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} = \frac{1}{4}\left[\left(0 + \frac{\cos n\pi}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= \frac{\cos n\pi - 1}{4n^2} = \frac{(-1)^n - 1}{4n^2}$$

Hence $a_n = \dfrac{-2}{4n^2} = \dfrac{-1}{2n^2}$, when n is odd

$= 0$, when n is even

Finally $b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} f(x) \sin nx \, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cdot \sin nx \, dx + \int_{0}^{\pi} \frac{\pi x}{4} \sin nx \, dx\right] = \frac{1}{\pi} \cdot \frac{\pi}{4} \int_{0}^{\pi} x \sin nx \, dx$$

$$= \frac{1}{4}\left[x\left(\frac{-\cos nx}{n}\right) - 1 \cdot \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{1}{4}\left[\left(\frac{-\pi \cos nx}{n} + 0\right) - (0 + 0)\right]$$

$$= \frac{-\pi \cos n\pi}{4n} = \frac{-\pi(-1)^n}{4n}$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$f(x) = \frac{\pi^2}{16} + \sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n - 1}{4n^2}\right)\cos nx - \frac{(-1)^n \pi}{4n} \sin nx\right]$$

$$= \frac{\pi^2}{16} - \frac{1}{2}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right)$$

$$+ \frac{\pi}{4}\left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right) \quad \ldots(2)$$

Deduction : At $x = 0$, $f(x)$ is continuous. Hence the Fourier series converges to $f(0)$ at $x = 0$. Putting $x = 0$ in (2), we obtain

$$f(0) = 0 = \frac{\pi^2}{16} - \frac{1}{2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right)$$

(*i.e.*) $\dfrac{\pi^2}{8} = 1 + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

Ex. 12. Obtain the Fourier series in $(-\pi, \pi)$ for the function

$f(x) = 0$, for $-\pi < x < 0$

$= \sin x$, for $0 < x < \pi$

Hence deduce that $\dfrac{1}{1 \cdot 3} - \dfrac{1}{1 \cdot 5} + \dfrac{1}{5 \cdot 7} - \ldots = \dfrac{1}{4}(\pi - 2)$ **[JNTU 2001]**

Sol. The Fourier series of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \text{...(1)}$$

Then $\quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \cdot dx + \int_{0}^{\pi} \sin x\, dx \right]$

$$= \frac{1}{\pi} \left[0 + (-\cos x)\right]_0^{\pi} = \frac{1}{\pi} (-\cos\pi + \cos 0) = \frac{1}{\pi}(1+1) = \frac{2}{\pi}$$

and $\quad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \cdot \cos nx\, dx + \int_{0}^{\pi} \sin x \cos nx\, dx \right]$$

$$= \frac{1}{2\pi} \int_0^{\pi} 2 \sin x \cos nx\, dx = \frac{1}{2\pi} \int_0^{\pi} \left[\sin (n+1)x + \sin (1-n)x\right] dx$$

$$= \frac{1}{2\pi} \int_0^{\pi} \left[\sin (n+1)x - \sin (n-1)x\right] dx$$

$$= \frac{1}{2\pi} \left[\frac{-\cos (n+1)x}{n+1} + \frac{\cos (n-1)x}{n-1} \right]_0^{\pi}, \text{ where } n \neq 1$$

$$= \frac{1}{2\pi} \left[\frac{-\cos (n+1)\pi}{n+1} + \frac{\cos (n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= \frac{1}{2\pi} \left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1} \right]$$

$$= \frac{1}{2\pi} \left[\frac{1}{n+1}(1-(-1)^{n+1}) + \frac{1}{n-1}\left((-1)^{n-1} - 1\right) \right] \; (n \neq 1)$$

$\therefore \quad a_n = 0$, when n is odd *i.e.* $n = 3, 5, 7, \ldots$

and $\quad a_n = \frac{1}{2\pi} \left(\frac{-4}{n^2-1} \right) = \frac{-2}{\pi (n^2-1)}$, when n is even and $n \neq 1$

In case $n = 1$, we have

$$a_1 = \frac{1}{\pi} \int_0^{\pi} \sin x \cos x\, dx = \frac{1}{2\pi} \int_0^{\pi} \sin 2x\, dx$$

$$= \frac{1}{2\pi} \left(\frac{-\cos 2x}{2} \right)_0^{\pi} = \frac{-1}{4\pi} (\cos 2\pi - \cos 0) = \frac{-1}{4\pi}(1-1) = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{\pi} \sin x \cdot \sin x dx = \frac{1}{2\pi}\int_0^{\pi} 2\sin^2 x\, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi}(1-\cos 2x)\; dx = \frac{1}{2\pi}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\left(\pi - \frac{\sin 2\pi}{2}\right) - (0-0)\right] = \frac{1}{2\pi}\,(\pi) = \frac{1}{2}$$

If $n \neq 1$, we have

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\; dx = \frac{1}{\pi}\int_0^{\pi}[\cos(n-1)x - \cos(n+1)x]dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left\{\left(\frac{\sin(n-1)\pi}{n-1} - \frac{\sin\,(n+1)\pi}{n+1}\right) - (0-0)\right\} = \frac{1}{2\pi}\,(0-0) = 0$$

Substituting the values of a_0, a_1, a_n, b_1 and b_n in (1), we get

$$f(x) = \frac{1}{\pi} - \frac{2}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \dots\right) + \frac{1}{2}\sin x$$

$$= \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{(2n)^2-1} + \frac{1}{2}\sin x \qquad \dots(2)$$

Deduction. Putting $x = \frac{\pi}{2}$ in (2), we obtain

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{(2n)^2-1} + \frac{1}{2}\sin\frac{\pi}{2}$$

$$\Rightarrow \sin\frac{\pi}{2} = 1 = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n+1)(2n-1)} + \frac{1}{2}$$

$$\text{or}\quad 1 - \frac{1}{2} - \frac{1}{\pi} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)\,(2n-1)}$$

$$\text{or}\quad \left(\frac{1}{2} - \frac{1}{\pi}\right)\left(\frac{\pi}{2}\right) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n+1)\,(2n-1)}$$

$$\text{or}\quad \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \dots = \frac{\pi-2}{4}$$

Ex. 13. Find the Fourier series of the following function

$$f(x) = -\cos x, \text{ when } -\pi < x < 0$$
$$= \cos x, \text{ when } 0 < x < \pi$$

Sol. The Fourier series of $f(x)$ in $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(1)$$

Then $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos x\, dx + \int_{0}^{\pi} \cos x\, dx\right]$

$$= \frac{1}{\pi}\left[(-\sin x)_{-\pi}^{0} + (\sin x)_{0}^{\pi}\right] = \frac{1}{\pi}(0+0) = 0$$

and $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos x \cdot \cos nx\, dx + \int_{0}^{\pi} \cos x \cdot \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{\pi}^{0} \cos(-u)\cos(-nu)\,(-du) + \int_{0}^{\pi} \cos x \cdot \cos nx\, dx\right].$$

(Putting $-u$ for x in the first integral)

$\therefore \quad a_n = \frac{1}{\pi}\left[\int_{\pi}^{0} \cos u \cos nu\, du + \int_{0}^{\pi} \cos x \cos nx\, dx\right]$

$$= \frac{1}{\pi}\left[-\int_{0}^{\pi} \cos x \cos nx\, dx + \int_{0}^{\pi} \cos x \cos nx\, dx\right] \qquad \left[\because \int_{a}^{b} f(x)\,dx = \int_{a}^{b} f(u)\,du\right]$$

$$= 0$$

Finally $\; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos x \sin nx\, dx + \int_{0}^{\pi} \cos x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\int_{\pi}^{0} \cos(-u)\sin n(-u)\,(-du) + \int_{0}^{\pi} \cos x \sin nx\, dx\right],$$

on putting $x = -u$ in the first integral

$\therefore \quad b_n = \frac{1}{\pi}\left[-\int_{\pi}^{0} \cos u \sin nu\, du + \int_{0}^{\pi} \cos x \sin nx\, dx\right]$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} \cos u \sin nu\, du + \int_{0}^{\pi} \cos x \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{0}^{\pi} \cos x \sin nx\, dx + \int_{0}^{\pi} \cos x \sin nx\, dx\right] = \frac{1}{\pi}\int_{0}^{\pi} 2\cos x . \sin nx\, dx$$

$$=\frac{1}{\pi}\int_0^\pi\left[\sin(n+1)x+\sin(n-1)x\right]dx$$

$$=\frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1}-\frac{\cos(n-1)x}{n-1}\right]_0^\pi,\ \text{where } n\neq 1$$

$$=\frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1}-\frac{\cos(n-1)\pi}{n-1}+\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$$=\frac{1}{\pi}\left[\frac{\cos n\pi}{n+1}+\frac{\cos n\pi}{n-1}+\frac{2n}{n^2-1}\right],$$

on expanding $\cos(n+1)\pi$ and $\cos(n-1)\pi$

$$\therefore\quad b_n=\frac{1}{\pi}\left[\frac{2n}{n^2-1}\cos n\pi+\frac{2n}{n^2-1}\right]$$

$$=\frac{2n}{\pi(n^2-1)}(1+\cos n\pi)=\frac{2n\,(1+(-1)^n)}{\pi(n^2-1)},\ n\neq 1$$

$\therefore\ b_n=0$, when n is odd *i.e.* $n=3, 5, 7, \ldots$

and $b_n=\dfrac{4n}{\pi(n^2-1)}$, or $\dfrac{4n}{\pi\,(n+1)\,(n-1)}$, when n is even.

When $n=1$, we have

$$b_1=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin x\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\cos x)\sin x\,dx+\int_0^\pi\cos x\sin x\,dx\right]$$

$$=\frac{1}{2\pi}\left[-\int_{-\pi}^{0}2\sin x\cos x\,dx+\int_0^\pi 2\sin x\cos x\,dx\right]$$

$$=\frac{1}{2\pi}\left[-\int_{-\pi}^{0}\sin 2x\,dx+\int_0^\pi\sin 2x\,dx\right]=\frac{1}{2\pi}\left[\left(\frac{\cos 2x}{2}\right)_{-\pi}^{0}+\left(-\frac{\cos 2x}{2}\right)_0^\pi\right]$$

$$=\frac{1}{2\pi}\left[\frac{1}{2}(1-1)=\frac{1}{2}(1-1)\right]=\frac{1}{2\pi}(0-0)=0$$

Substituting the values of a_0, a_n, b_n in (1), we get

$$f(x)=\frac{4}{\pi}\left[\frac{2}{1\cdot 3}\sin 2x+\frac{4}{3\cdot 5}\sin 4x+\frac{6}{5\cdot 7}\sin 6x+\ldots\right]$$

which is the required Fourier series.

Ex. 14: Given :

$$f(x)=1+\frac{2x}{\pi},\ \text{when } -\pi\le x\le 0$$

$$=1-\frac{2x}{\pi},\ \text{when } 0\le x\le\pi$$

Show that $f(x)=\dfrac{8}{\pi^2}\left(\dfrac{\cos x}{1^2}+\dfrac{\cos 3x}{3^2}+\dfrac{\cos 5x}{5^2}+\ldots\right)$

Deduce from this that $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots=\dfrac{\pi^2}{8}$ **[JNTU 2000S]**

Sol. Let $f(x) \simeq \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ ...(1)

Then $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \left[\int_{-\pi}^{0} \left(1 + \frac{2x}{\pi}\right) dx + \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) dx \right]$

$$= \frac{1}{\pi} \left[\left(x + \frac{x^2}{\pi}\right)_{-\pi}^{0} + \left(x - \frac{x^2}{\pi}\right)_0^{\pi} \right] = \frac{1}{\pi} [0 + 0] = 0$$

and $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} \left(1 + \frac{2x}{\pi}\right) \cos nx\, dx + \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \left\{ \left[\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \frac{2}{\pi} \left(x \left(\frac{\sin nx}{n}\right) - 1 \left(\frac{-\cos nx}{n^2}\right) \right)_{-\pi}^{0} \right] \right.$$

$$\left. + \left[\left(\frac{\sin nx}{n}\right)_0^{\pi} - \frac{2}{\pi} \left(x \left(\frac{\sin nx}{n}\right) - 1 \cdot \left(\frac{-\cos nx}{n^2}\right) \right)_0^{\pi} \right] \right\}$$

$$= \frac{1}{\pi} \left\{ \left[0 + \frac{2}{\pi} \left(\frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right) \right] + \left[0 - \frac{2}{\pi} \left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2} \right) \right] \right\}$$

$$= \frac{2}{\pi^2} \left[\frac{1}{n^2} - \frac{\cos n\pi}{n^2} - \frac{\cos n\pi}{n^2} + \frac{1}{n^2} \right]$$

$$= \frac{2}{\pi^2} \left[\frac{2}{n^2} - \frac{2 \cos n\pi}{n^2} \right] = \frac{4}{\pi^2 n^2} (1 - \cos n\pi)$$

$$= \frac{4[1 - (-1)^n]}{\pi^2 n} = \frac{8}{\pi^2 n^2} \text{ when } n \text{ is odd}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} \left(1 + \frac{2x}{\pi}\right) \sin nx\, dx + \int_0^{\pi} \left(1 - \frac{2x}{\pi}\right) \sin nx\, dx \right]$$

$$= \frac{1}{\pi} \left\{ \left[\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \frac{2}{\pi} \left[x \left(\frac{-\cos nx}{n}\right) - 1 \cdot \left(\frac{-\sin nx}{n^2}\right)_{-\pi}^{0} \right] \right] \right.$$

$$\left. + \left[\left(\frac{-\cos nx}{n}\right)_0^{\pi} - \frac{2}{\pi} \left[x \left(\frac{-\cos nx}{n}\right) - 1 \cdot \left(\frac{-\sin nx}{n^2}\right) \right]_0^{\pi} \right] \right\}$$

$$= \left[\frac{-1}{n} + \frac{\cos n\pi}{n} + \frac{2}{\pi} \left(\frac{-\pi \cos n\pi}{n} \right) \right] + \left[\frac{-\cos n\pi}{n} + \frac{1}{n} - \frac{2}{\pi} \left(\frac{-\pi \cos n\pi}{n} \right) \right]$$

$$= 0$$

Substituting the values of a_0, a_n, b_n in (1), we get

$$f(x)=\frac{8}{\pi^2}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\dots\right] \quad \dots(2)$$

Hence the result.

Deduction. Putting $x = 0$ in (2), we get

$$f(0)=\frac{8}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right)$$

$$\Rightarrow 1=\frac{8}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right) \text{ or } \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8}.$$

Ex. 15: Find the Fourier series of the periodic function defined as

$$f(x)=\begin{cases}-\pi & , \quad -\pi<x<0 \\ x & , \quad 0<x<\pi\end{cases}$$

Hence deduce that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots=\frac{\pi^2}{8}$ **[JNTU 2004, 2006S (Set No. 2)]**

Sol. Let $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n \sin nx \quad \dots(1)$

Then
$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\,dx+\int_0^{\pi} x\,dx\right]$$

$$=\frac{1}{\pi}\left[(-\pi)(x)_{-\pi}^{0}+\left(\frac{x^2}{2}\right)_0^{\pi}\right]=\frac{1}{\pi}\left[-\pi^2+\frac{\pi^2}{2}\right]=\frac{1}{\pi}\left(\frac{-\pi^2}{2}\right)=\frac{-\pi}{2}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\cos nx\,dx+\int_0^{\pi} x\cos nx\,dx\right]$$

$$=\frac{1}{\pi}\left[(-\pi)\left(\frac{\sin nx}{n}\right)_{-\pi}^{0}+\left(x.\frac{\sin nx}{n}+\frac{\cos nx}{n^2}\right)_0^{\pi}\right]$$

$$=\frac{1}{\pi}\left[0+\frac{1}{n^2}\cos n\pi-\frac{1}{\pi n^2}\right]$$

$$=\frac{1}{\pi n^2}(\cos n\pi-1)=\frac{1}{\pi n^2}\left[(-1)^n-1\right]$$

$$\therefore \quad a_1=\frac{-2}{1^2.\pi},\ a_2=0,\ a_3=\frac{-2}{3^2.\pi},\ a_4=0, a_5=\frac{-2}{5^2.\pi},\dots$$

and $\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-\pi)\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right]$

$$= \frac{1}{\pi}\left[\pi\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left(-x.\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right] = \frac{1}{n}(1-2\cos n\pi)$$

$\therefore \quad b_1 = 3,\ b_2 = \frac{-1}{2},\ b_3 = 1,\ b_4 = \frac{-1}{4}$, and so on

Substituting the values of a_0, a_n and b_n in (1), we get

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + \left(3\sin x - \frac{\sin 2x}{2} + \frac{3\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\right) \quad \ldots(2)$$

Deduction. Putting $x = 0$ in (2), we obtain

$$f(0) = -\frac{\pi}{4} - \frac{2}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \quad \ldots(3)$$

Now $f(x)$ is discontinuous at $x = 0$

$f(0-0) = -\pi$ and $f(0+0) = 0$

$\therefore \quad f(0) = \frac{1}{2}[f(0-0) + f(0+0)] = \frac{-\pi}{2}$

Now (3) becomes

$$\frac{-\pi}{2} = \frac{-\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \text{ (or) } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

Ex. 16: Find the Fourier series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} 0, & \text{for } -\pi \le x \le 0 \\ x^2, & \text{for } 0 \le x \le \pi \end{cases}$$

Sol. The Fourier series of $f(x)$ in $[-\pi, \pi]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \quad \ldots(1)$$

Then $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_0^{\pi} x^2\,dx\right] = \frac{1}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{\pi^2}{3}$

and $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0.\cos nx\,dx + \int_0^{\pi} x^2\cos nx\,dx\right] = \frac{1}{\pi}\int_0^{\pi} x^2\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi} \cdot \frac{2\pi}{n^2} \cos n\pi = \frac{2}{n^2}(-1)^n, \text{ for } n = 1, 2, 3, \ldots$$

Finally $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{\pi} x^2 \sin nx \, dx$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi^2}{n} \cos n\pi + \frac{2}{n^3}(\cos n\pi - 1)\right]$$

$$= \frac{-\pi}{n}(-1)^n + \frac{2}{\pi n^3}\left[(-1)^n - 1\right]$$

Substituting the values of a's and b's in (1), we get

$$f(x) = \frac{\pi^2}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx + \sum_{n=1}^{\infty}\left[\frac{\pi}{n}(-1)^{n+1} + \frac{2}{\pi n^3}(-1)^n - 1\right] \sin nx$$

Ex. 17: Obtain the Fourier series in $[-\pi, \pi]$ for the function

$$f(x) = \begin{cases} \dfrac{-1}{2}(\pi + x), & \text{for } -\pi \le x \le 0 \\ \dfrac{1}{2}(\pi - x), & \text{for } 0 \le x \le \pi \end{cases}$$

Sol. Let the function $f(x)$ be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \qquad \ldots(1)$$

Then $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} \frac{-1}{2}(\pi + x)\, dx + \int_0^{\pi} \frac{1}{2}(\pi - x)\, dx\right]$

$$= \frac{1}{2\pi}\left[-\left(\pi x + \frac{x^2}{2}\right)_{-\pi}^{0} + \left(\pi x - \frac{x^2}{2}\right)_0^{\pi}\right] = \frac{1}{2\pi}\left[-\pi^2 + \frac{\pi^2}{2} + \pi^2 - \frac{\pi^2}{2}\right] = 0$$

and $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} \frac{-1}{2}(\pi + x) \cos nx \, dx + \int_0^{\pi} \frac{1}{2}(\pi - x) \cos nx \, dx\right]$$

$$= \frac{-1}{2\pi}\left[\int_{-\pi}^{0} \pi \cos nx \, dx + \int_{-\pi}^{0} x \cos nx \, dx - \int_0^{\pi} \pi \cos nx \, dx + \int_0^{\pi} x \cos nx \, dx\right]$$

$$= \frac{-1}{2\pi}\left\{\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_{-\pi}^{0} - \pi\left(\frac{\sin nx}{n}\right)_0^{\pi} + \left[x\left(\frac{\sin nx}{n}\right) + \frac{\cos nx}{n^2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left[\frac{1-\cos n\pi}{n^2} + \frac{\cos n\pi - 1}{n^2}\right] = 0$$

Finally $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{-1}{2}(\pi+x)\sin nx\,dx + \int_{0}^{\pi}\frac{1}{2}(\pi-x)\sin nx\,dx\right]$$

$$= \frac{-1}{2\pi}\left[\int_{-\pi}^{0}\pi\sin nx\,dx + \int_{-\pi}^{0}\pi\sin nx\,dx - \int_{0}^{\pi}\pi\sin nx\,dx + \int_{0}^{\pi}\pi\sin nx\,dx\right]$$

$$= \frac{-1}{2\pi}\left\{\pi\left(\frac{-\cos nx}{n}\right)_{-\pi}^{0} + \left[x\left(\frac{-\cos nx}{n}\right) + \frac{\cos nx}{n^2}\right]_{-\pi}^{0}\right.$$

$$\left. + \pi\left(\frac{\cos nx}{n}\right)_{0}^{\pi} + \left[x\left(\frac{-\cos nx}{n}\right) + \frac{\cos nx}{n^2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-\pi}{n}(1-\cos n\pi) + \frac{1}{n^2} - \left(\frac{\pi\cos n\pi}{n} + \frac{\cos n\pi}{n^2}\right)\right.$$

$$\left. + \frac{\pi}{n}(\cos n\pi - 1) + \left[\frac{-\pi\cos n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right]\right\}$$

$$= \frac{-1}{2\pi}\left(\frac{-2\pi}{n}\right) = \frac{1}{n}$$

Substituting the values of a's and b's in (1), we get

$$f(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n} = \sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots$$

Ex. 18: A sinusoidal voltage $E \sin wt$ is passed through a half-wave rectifier which clips the negative portion of the wave. Develop the resulting periodic function

$$U(t) = 0, \text{ when } -\frac{T}{2} < t < 0$$
$$= E\sin wt, \text{ when } 0 < t < \frac{T}{2}$$

and $T = \frac{2\pi}{w}$ in a Fourier series.

Sol. Let $U(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + \sum_{n=1}^{\infty} b_n \sin\frac{2n\pi t}{T}$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nwt + \sum_{n=1}^{\infty} b_n \sin nwt \quad \left(\because T = \frac{2\pi}{w}\right) \qquad \ldots (1)$$

Then $a_0 = \frac{2}{T}\int_{-T/2}^{T/2} U(t)\,dt$

$$= \frac{2}{T}\left[\int_{-T/2}^{0} 0\,dt + \int_{0}^{T/2} E\sin wt\,dt\right] = \frac{2}{T}\int_{0}^{T/2} E\sin wt\,dt$$

$$= \frac{2}{T}\left(\frac{-E}{w}\cos wt\right)_0^{T/2} = \frac{-2E}{wT}(\cos\pi - 1) = \frac{2E}{\pi} \qquad (\because wT = 2\pi)$$

and $$a_n = \frac{2}{T}\int_{-T/2}^{T/2} U(t)\cos nwt\,dt = \frac{2}{T}\int_0^{T/2} E\sin wt\cos nwt\,dt$$

$$= \frac{E}{T}\int_0^{T/2}\left[\sin(1+n)wt + \sin(1-n)wt\right]dt$$

$$= \frac{E}{T}\left[\frac{-\cos(1+n)wt}{(1+n)w} - \frac{\cos(1-n)wt}{(1-n)w}\right]_0^{T/2}, \text{ for } n \neq 1$$

$$= -\frac{E}{T}\left[\frac{\cos(1+n)\pi}{(1+n)w} + \frac{\cos(1-n)\pi}{(1-n)w} - \frac{1}{(1+n)w} - \frac{1}{(1-n)w}\right]$$

$$= \frac{-E}{wT}\left[\frac{-1}{(1+n)} - \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right], \text{ when } n \text{ is even}$$

$$= \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right) = \frac{-2E}{\pi(n^2-1)}, \text{ when } n \text{ is even}$$

If n is odd ($n \neq 1$), $a_n = 0$
In case $n = 1$, we have

$$a_1 = \frac{2}{T}\int_0^{T/2} E\sin wt\cos wt\,dt = \frac{E}{T}\int_0^{T/2}\sin 2wt\,dt$$

$$= \frac{E}{2wT}(\cos wT - \cos 0) = \frac{-E}{4\pi}(\cos 2\pi - \cos 0) = 0$$

and $$b_1 = \frac{2}{T}\int_0^{T/2} E\sin wt\cos wt\,dt = \frac{E}{T}\int_0^{T/2}(1-\cos 2wt)\,dt$$

$$= \frac{E}{T}\left[t - \frac{\sin 2wt}{2w}\right]_0^{T/2} = \frac{E}{2}, (\text{since } \sin wT = \sin 2\pi = 0)$$

If $n \neq 1$, then we have

$$b_n = \frac{2}{T}\int_0^{T/2} E\sin wt\sin nwt\,dt \quad (\because 2\pi = Tw)$$

$$= \frac{E}{T}\int_0^{T/2}\left[\cos(n-1)wt - \cos(n+1)wt\right]dt = 0$$

Substituting the values of a's and b's in (1), we get

$$\therefore \quad U(t) = \frac{E}{\pi} + \frac{E}{2}\sin wt - \frac{2E}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{1}{n^2-1}\cos n\omega t$$

Ex. 19: The intensity of an alternating current after passing through a rectifier is given by

$$i(x) = \begin{cases} I_0 . \sin x, & \text{for } 0 \le x \le \pi \\ 0, & \text{for } \pi \le x \le 2\pi \end{cases}$$

Where I_0 is maximum current and the period is 2π. Express $i(x)$ as a Fourier series. **[JNTU 2002, 2005S, 2006S (Set No. 1)]**

Sol. The Fourier series of $i(x)$ in $[0, 2\pi]$ is given by

$$i(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad \ldots(1)$$

Then $a_0 = \frac{1}{\pi}\left[\int_0^{\pi} I_0 \sin x\, dx + \int_{\pi}^{2\pi} 0.dx\right]$

$$= \frac{I_0}{\pi}\int_0^{\pi} \sin x\, dx = \frac{-I_0}{\pi}(\cos x)_0^{\pi} = \frac{-I_0}{\pi}(-1-1) = \frac{2I_0}{\pi}$$

and $a_n = \frac{1}{\pi}\left[\int_0^{\pi} I_0 \sin x . \cos nx\, dx + \int_{\pi}^{2\pi} 0 . \cos nx\, dx\right]$

$$= \frac{I_0}{\pi}\int_0^{\pi} \sin x \cos nx\, dx = \frac{I_0}{2\pi}\int_0^{\pi} 2 \sin x \cos nx\, dx$$

$$= \frac{I_0}{2\pi}\int_0^{\pi} [\sin(n+1)x + \sin(1-n)x]\, dx$$

$$= \frac{I_0}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{I_0}{2\pi}\frac{\left[2(-1)^n + 2\right]}{\left(n^2-1\right)} = \frac{I_0}{\pi(n^2-1)}\left[(-1)^n + 1\right] \text{ for } n \ne 1$$

$\therefore \quad a_n = \frac{2I_0}{\pi(n^2-1)}$, when n is even

$= 0$, when n is odd $\quad (n \ne 1)$

$\therefore \quad a_1 = \frac{1}{\pi}\left[\int_0^{\pi} I_0 \sin x . \cos x\, dx + \int_{\pi}^{2\pi} 0 . \cos x\, dx\right]$

$$= \frac{I_0}{2\pi}\int_0^{\pi} \sin 2x\, dx = \frac{I_0}{2\pi}\left[\frac{-\cos 2x}{2}\right]_0^{\pi} = 0$$

and $\quad b_1 = \frac{1}{\pi}\int_0^{\pi} I_0 \sin^2 x\, dx = \frac{I_0}{2}$

Finally $\quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} I_0 \sin x.\sin nx\, dx, \ (n \neq 1)$

$$= \frac{I_0}{2\pi}\int_0^{\pi}\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{I_0}{2\pi}\left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1}\right]_0^{\pi}$$

$$= 0, \quad \text{for all} \quad n \neq 1$$

Substituting the values of a's and b's in (1), we get

$$i(x) = \frac{I_0}{\pi} + \frac{I_0}{2}\sin x - \frac{2I_0}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1} \quad (\because n \text{ is even})$$

EXERCISE 10 (A)

1. (*a*) Define a periodic function. **[JNTU 2003]**

   (*b*) Write the Dirichlet's conditions for the existence of Fourier series of a function $f(x)$ in the interval $(\alpha, \alpha + 2\pi)$. **[JNTU 2003 (Set No. 2), 2004, 2005]**

2. Obtain the Fourier series for the function.

   (*i*) $e^x - 1$ in $(0, 2\pi)$ (*ii*) e^{-x} in the interval $(0, 2\pi)$ [**Hint :** Refer Solved Ex. 4]

   (*iii*) e^{ax} in $(-\pi, \pi)$ **[JNTU 2001]** (*iv*) e^{ax} in $(0, 2\pi)$

3. Show that in the range $-\pi$ to π, $x + x^2$ as a Fourier series is

   $$x + x^2 = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - ....\right) + 2\left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - ...\right)$$

   Hence show that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$ **[JNTU 2003 (Set No. 3)]**

4. (*i*) Find the Fourier series of $\pi^2 - x^2$ in $(-\pi, \pi)$.

   (*ii*) Find the Fourier series of $f(x) = \pi - x$ in $(0, 2\pi)$

5. If $f(x) = \frac{\pi - x}{2}$ in the range 0 to 2π, show that $f(x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + ...$

   Hence deduce that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = \frac{\pi}{4}$

6. Express $f(x) = x + |x|$ as Fourier series in $(-\pi, \pi)$

   [**Hint :** $f(x) = x - x = 0$ when $-\pi \leq x \leq 0$
   $= x + x = 2x$ when $0 \leq x \leq \pi$]

7. Expand $f(x) = x \cos x$, $0 < x < 2\pi$ as a Fourier series

8. Express $f(x) = x(2\pi - x)$ as Fourier series in $(0, 2\pi)$

   Deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ... = -\frac{\pi^2}{6}$

9. Given : $f(x) = \frac{\pi}{2} + x$, for $-\pi < x < 0 = \frac{\pi}{2} - x$, for $0 < x < \pi$

Show that $f(x) = \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$

10. If $f(x) = 0$, for $-\pi < x \le 0$

$= x$, for $0 \le x \le \pi$

Prove that $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right)$

Hence show that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$

11. Find the Fourier series of the following function :

$f(x) = -1 + x$, for $-\pi < x < 0$

$= 1 + x$, for $0 < x < \pi$

Hence prove that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$

12. Find the Fourier series expansion for $f(x)$, if

$f(x) = x$, when $0 \le x \le \pi$

$= 2\pi - x$, when $\pi \le x \le 2\pi$

Hence deduce that $1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$ **[JNTU June 2003]**

Or Obtain the Fourier series for $f(x) = \pi - |x - \pi|$ in $(0, 2\pi)$

13. If $f(x) = x(\pi - x)$, when $0 < x < \pi$

$= -\pi(\pi - x)$, when $\pi < x < 2\pi$

Prove that $f(x) = \frac{8}{\pi}\left(\sin x + \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x + \ldots\right)$

14. Find the Fourier series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} -x^2, -\pi \le x < 0 \\ x^2, 0 \le x \le \pi \end{cases}$$

ANSWERS

2. (i) $e^x - 1 = \frac{e^{2\pi} - 1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2+1} - \sum_{n=1}^{\infty} \frac{n\sin nx}{n^2+1}\right] - 1$

(ii) $e^{-x} = \frac{1 - e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty} \frac{\cos nx}{n^2+1} + \sum_{n=1}^{\infty} \frac{n\sin nx}{n^2+1}\right]$

(iii) $e^{ax} = \frac{2\sinh a\pi}{\pi}\left[\frac{1}{2a} + a\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{a^2+n^2} + \sum_{n=1}^{\infty} \frac{(-1)^n n\sin nx}{a^2+n^2}\right]$

(iv) $e^{ax} = \frac{e^{2a\pi} - 1}{2} + \frac{ae^{2a\pi} - 1}{\pi}\sum_{n=1}^{\infty} \frac{\cos nx}{a^2+n^2} + \frac{1 - e^{2a\pi}}{\pi} + \sum_{n=1}^{\infty} \frac{n}{a^2+n^2}\sin nx$

ODS

4. $\pi^2 - x^2 = \frac{2}{3}\pi^2 - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$

6. $f(x) = \pi - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] + 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \ldots\right]$

7. $f(x) = \pi \cos x - \frac{1}{2}\sin x - 2 \sum_{n=2,3\ldots}^{\infty} \frac{n}{n^2 - 1} \sin nx$

8. $f(x) = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{-4}{n^2} \cos nx$

11. $f(x) = \frac{2}{\pi}(\pi + 2)\sin x - \frac{2}{2}\sin 2x + \frac{2(\pi+2)}{3\pi}\sin 3x - \frac{2}{4}\sin 4x + \frac{2(\pi+2)}{5\pi}\sin 5x - \ldots$

12. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right)$

14. $f(x) = 2\left(\pi - \frac{4}{\pi}\right)\sin x - \pi \sin 2x + \frac{2}{3}\left(\pi - \frac{4}{9\pi}\right)\sin 3x - \frac{\pi}{2}\sin 4x + \ldots$

10.6 EVEN AND ODD FUNCTIONS

A function $f(x)$ is said to be even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$

For example x^2, $x^4 + x^2 + 1$, $e^x + e^{-x}$, $\cos x$, $\sec x$ are all even functions of x, while x, x^3, $x^5 + 2x^3 + 3$, $\sin x$, $\operatorname{cosec} x$, $\tan x$ are all odd functions.

Graphically an even function is symmetrical about the y-axis and an odd function is symmetrical about the origin.

The product of two even or two odd functions will be an even function while the product of an even function and an odd function will be an odd function.

We shall be frequently using the following property of definite integrals :

$$\int_{-a}^{a} f(x)\, dx = 0, \text{ when } f(x) \text{ is an odd function.}$$

$$= 2\int_{0}^{a} f(x)\, dx, \text{ when } f(x) \text{ is an even function.}$$

Note : It is desirable to consider the even or odd nature of a function $f(x)$ when we are dealing with domain of definition in the form $(-l, l)$, $(-\pi, \pi)$ etc.

10.7 FOURIER SERIES FOR EVEN AND ODD FUNCTIONS

We know that a function $f(x)$ defined in $(-\pi, \pi)$ can be represented by the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$

Case I. When $f(x)$ is an even function

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x)\, dx$$

Since **cos *nx*** is an even function, $f(x)$ **cos *nx*** is also an even function. Hence

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\, dx$$, since the integrand is even.

Again since **sin *nx*** is an odd function, $f(x)$ **sin *nx*** is an odd function

$$\therefore b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = 0$$, since the integrand is odd.

Thus, if a function $f(x)$ is even in $(-\pi, \pi)$, its Fourier series expansion contains only cosine terms.

Hence we have the following :

If $f(x)$ is defined in $[-\pi, \pi]$ and $f(x)$ is an even function, $f(x)$ can be expanded as a series in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \cos nx\, dx,\ n = 0, 1, 2 \ldots.$

Case II. When $f(x)$ is an odd function in $(-\pi, \pi)$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = 0 \text{ since } f(x) \text{ is odd}$$

Since **cos *nx*** is an even function, $f(x)$ **cos *nx*** is an odd function and hence

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = 0$$, since the integrand is odd.

Again since **sin *nx*** is an odd function, $f(x)$ **sin *nx*** is an even function.

$$\therefore b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx\, dx$$, since the integrand is even.

Thus, if a function $f(x)$ defined in $(-\pi, \pi)$ is odd, its Fourier expansion contains only sine terms.

Hence if $f(x)$ is an odd function defined in $[-\pi, \pi]$, $f(x)$ can be expanded as a series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx\, dx$

EXAMPLES

Ex. 1: Express $f(x) = x$ as a Fourier series in $(-\pi, \pi)$. **[JNTU 2002S]**

Sol. Since $f(-x) = -x = -f(x)$

$\therefore$ $f(x)$ is an odd function in $(-\pi, \pi)$.

Hence in its Fourier series expansion, the cosine terms are absent and only sine terms are present.

$$\therefore x = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{where} \quad b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{-2}{n}\cos n\pi = (-1)^{n+1}\frac{2}{n} \qquad (\because \sin n\pi = 0)$$

$$\therefore \; x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nx$$

$$= 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right)$$

which is the required Fourier series.

Ex. 2: Expand the function $f(x) = x^2$ as a Fourier series in $[-\pi, \pi]$.

or, Proove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}$. **[JNTU 2003 (Set No. 2)]**

Hence deduce that

(*i*) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}$ (*ii*) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$

(*iii*) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$ **[JNTU 2003S (Set No. 2)]**

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$,

$\therefore$ $f(x)$ is an even function in $[-\pi, \pi]$.

Hence in its Fourier series expansion, the sine terms are absent.

$$\therefore x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(1)$$

$$\text{where} \quad a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 \, dx = \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2\pi^2}{3} \qquad \ldots(2)$$

$$\text{and} \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + 2\pi\,\frac{\cos nx}{n^2} + 2\cdot 0\right]$$

$$= \frac{4\cos n\pi}{n^2} = \frac{4}{n^2}(-1)^n \qquad ...(3)$$

Substituting the values of a_0 and a_n from (2) and (3) in (1), we get

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$ [JNTU 2003 (Set No. 2)]

$$= \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} - 4\left(\cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \dots\right) \qquad ...(4)$$

Deductions.

(*i*) Putting $x = 0$ in (4), we get

$$0 = \frac{\pi^2}{3} - 4\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right)$$

$$\Rightarrow 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12} \qquad ...(A)$$

(*ii*) Putting $x = \pi$ in (4), we have

$$\pi^2 = \frac{\pi^2}{3} - 4\left(\cos\pi - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \frac{\cos 4\pi}{4^2} + \dots\right)$$

$$= \frac{\pi^2}{3} - 4\left(-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \dots\right)$$

$$= \frac{\pi^2}{3} + 4\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots\right)$$

$$\Rightarrow 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6} \qquad ...(B)$$

(*iii*) Adding (A) and (B) and dividing it by 2, we get the required result as

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$

Ex. 3: Obtain the Fourier Series for the function $f(x) = |x|$ in $-\pi < x < \pi$ and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$ [JNTU 2003S (Set No. 4)]

Sol. Since $f(-x) = x = f(x)$, therefore $f(x) = |x|$ is an even function. Hence the Fourier series will consist of cosine terms only.

$$\therefore f(x) = |x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad ...(1)$$

where $a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx = \frac{2}{\pi}\int_0^\pi |x|\,dx = \frac{2}{\pi}\int_0^\pi x\,dx$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^\pi = \frac{2}{\pi}\left(\frac{\pi^2}{2}\right) = \pi \qquad ...(2)$$

and $a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi |x|\cos nx\,dx = \frac{2}{\pi}\int_0^\pi x\cos nx\,dx$

$$= \frac{2}{\pi}\left[x\cdot\frac{\sin nx}{n} - 1\cdot\left(\frac{-\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$\therefore a_n = 0$, if n is even

$$= \frac{-4}{\pi n^2}, \text{ if } n \text{ is odd} \qquad ...(3)$$

Substituting the values of a_0 and a_n from (2) and (3) in (1), we get

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) \qquad ...(4)$$

Deduction

When $x = 0$, $|x| = |0| = 0$

$\therefore$ Putting $x = 0$ in (4), we have

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

Ex. 4: Show that for $-\pi < x < \pi$,

$$\sin ax = \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2 - a^2} - \frac{2\sin 2x}{2^2 - a^2} + \frac{3\sin 3x}{3^2 - a^2} - \ldots\right] \text{ (}a\text{ is not an integer)}$$

[JNTU 2004S (Set No. 2, 4), 2005S (Set No. 2)]

Sol. As $\sin ax$ is an odd function, its Fourier series expansion will consist of sine terms only.

$$\therefore \sin ax = \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(1)$$

where $b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi \sin ax \sin nx\,dx$

$$= \frac{1}{\pi}\int_0^\pi [\cos(a-n)x - \cos(a+n)x]\,dx \quad [\because 2\sin A\sin B = \cos(A-B) - \cos(A+B)]$$

$$\therefore b_n = \frac{1}{\pi}\left[\frac{\sin(a-n)x}{a-n} - \frac{\sin(a+n)x}{a+n}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{\sin(a-n)\pi}{a-n} - \frac{\sin(a+n)\pi}{a+n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin a\pi\,\cos n\pi - \cos\pi\,\sin n\pi}{a-n} - \frac{\sin a\pi\,\cos n\pi + \cos a\pi\,\sin n\pi}{a+n}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin a\pi\cos n\pi}{a-n} - \frac{\sin a\pi\cos n\pi}{a+n}\right] \quad [\because \sin n\pi = 0]$$

$$= \frac{1}{\pi}\sin a\pi\cos n\pi\left(\frac{1}{a-n}-\frac{1}{a+n}\right) = \frac{1}{\pi}\sin a\pi\ (-1)^n\left(\frac{a+n-a+n}{a^2-n^2}\right)$$

$$= \frac{(-1)^n 2n}{\pi\,(a^2-n^2)}\,\sin a\pi$$

Substituting these values in (1), we get

$$\sin ax = \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^n}{a^2-n^2}\sin nx$$

$$= \frac{2\sin a\pi}{\pi}\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{n^2-a^2}\sin nx\right]$$

$$= \frac{2\sin a\pi}{\pi}\left(\frac{\sin x}{1^2-a^2}-\frac{2\sin 2x}{2^2-a^2}+\frac{3\sin 3x}{3^2-a^2}-\dots\right)$$

(We note here that since a is not an integer, 1^2-a^2, 2^2-a^2, 3^2-a^2 etc. will not become zero).

Ex. 5: Find the Fourier series to represent the function

$$f(x) = x\sin x,\ -\pi < x < \pi$$

Hence deduce that $\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\frac{1}{7\cdot 9}+\dots=\frac{1}{4}(\pi-2)$ **[JNTU 2004S (Set No. 3)]**

Sol. Since $f(x) = x\sin x$ is an even function, $b_n = 0$

Let $x\sin x = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$. Then

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx = \frac{2}{\pi}\left[x(-\cos x) - 1\cdot(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}(-\pi\cos\pi) = 2$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\left[\sin(n+1)x + \sin(1-n)x\right]dx$$

$$= \frac{1}{\pi}\left\{x\left[\frac{-\cos(n+1)x}{n+1}-\frac{\cos(1-n)x}{1-n}\right] - 1\cdot\left[\frac{-\sin(n+1)x}{(n+1)^2}-\frac{\sin(1-n)x}{(1-n)^2}\right]\right\}_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(1-n)\pi}{1-n}\right)\right],\ (n\neq 1)$$

(Since the second term vanishes at both upper and lower limits)

$$\therefore\ a_n = -\left[\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1}\right],\ [\because \cos(-\theta) = \cos\theta]$$

$$= -\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}(-1)^2}{n-1}\right]$$

$$= -\left[\frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n+1}}{n-1}\right] = (-1)(-1)^{n+1}\left[\frac{n-1-n-1}{n^2-1}\right]$$

$$= \frac{2(-1)^{n+1}}{n^2-1},\ n \neq 1 \qquad ..(A)$$

If we put $n = 1$ then $a_1 = \infty$ and hence a_1 cannot be evaluated from (A).

Putting $n = 1$ in $a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\, dx$, we get

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} x \sin 2x\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1\cdot\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{-\pi\cos 2\pi}{2}\right) = \frac{-1}{2},$$

since the second term vanishes at both upper and lower portions.

$$\therefore\ x \sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2-1}\cos nx$$

$$= 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{(n-1)(n+1)}\cos nx$$

$$= 1 - \frac{1}{2}\cos x + \left(\frac{-2}{1\cdot 3}\cos 2x + \frac{2}{2\cdot 4}\cos 3x - \frac{2}{3\cdot 5}\cos 4x + \ldots\right) \qquad \ldots(B)$$

Deduction

Putting $x = \frac{\pi}{2}$ in (B), we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - 0 - \frac{2}{1\cdot 3}(-1) + \frac{2}{2\cdot 4}(0) + \frac{-2}{3\cdot 5}(1) + \ldots$$

$$\Rightarrow \frac{\pi}{2} = 1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \ldots$$

$$\Rightarrow \frac{\pi}{2} - 1 = \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \frac{2}{7\cdot 9} + \ldots$$

$$\Rightarrow \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \ldots = \frac{1}{4}(\pi - 2)$$

Ex. 6: Express $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ as a Fourier series in the interval $-\pi < x < \pi$.

Sol.

Since $f(-x) = \frac{\pi^2}{12} - \frac{(-x)^2}{4} = \frac{\pi^2}{12} - \frac{x^2}{4} = f(x)$

$\therefore f(x)$ is an even function and hence

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)dx = \frac{2}{\pi}\left[\frac{\pi^2}{12}x - \frac{x^3}{12}\right]_0^{\pi} = 0$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\cos nx\ dx$

Integrating by parts, we obtain

$$a_n = \frac{2}{\pi}\left[\left(\frac{\pi^2}{12} - \frac{x^2}{4}\right)\frac{\sin nx}{n} - \left(\frac{-2x}{4}\right)\left(\frac{-\cos nx}{n^2}\right) + \left(\frac{-1}{2}\right)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\frac{-1}{n^2}\right)\frac{\pi}{2}\cos n\pi = \frac{(-1)^{n+1}}{n^2}$$

Hence $f(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx = \cos x - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots$

Ex. 7: Find the Fourier series to represent the function

$f(x) = \sin x, -\pi < x < \pi$ **[JNTU Dec 2002 (Set No. 3)]**

Sol. Since $\sin x$ is an odd function, $a_0 = a_n = 0$

Let $f(x) = \Sigma b_n \sin nx$, where

$$b_n = \frac{2}{\pi}\int_0^{\pi}\sin x \sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi}[\cos(1-n)x - \cos(1+n)x]\, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right]_0^{\pi} \quad (n \neq 1)$$

$$= \frac{1}{\pi}\left[\frac{\sin(1-n)x}{1-n} - \frac{\sin(1+n)x}{1+n}\right] \quad (n \neq 1)$$

$$= 0 \ (n \neq 1)$$

If $n = 1$, then

$$b_1 = \frac{2}{\pi}\int_0^{\pi} \sin^2 x\, dx = \frac{1}{\pi}\int_0^{\pi}(1-\cos 2x)dx$$

$$= \frac{1}{\pi}\left(x - \frac{\sin 2x}{2}\right)_0^{\pi} = \frac{1}{\pi}(\pi - 0) = 1$$

$\therefore f(x) = b_1 \sin x = \sin x$

Ex. 8: Find the Fourier series to represent the function

$f(x) = |\sin x|, -\pi < x < \pi.$ **[JNTU 2004S (Set No.2), 2006S (Set No.3)]**

Sol. Since $|\sin x|$ is an even function,

$\therefore \quad b_n = 0$ for all n.

Let $f(x) = |\sin x| = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots (1)$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} |\sin x|\, dx = \dfrac{2}{\pi}\int_0^{\pi} \sin x\, dx = \dfrac{2}{\pi}(-\cos x)_0^{\pi}$ $\quad [\because \sin x = \sin x \text{ in } 0<x<\pi]$

$$= \frac{-2}{\pi}(-1-1) = \frac{4}{\pi}$$

and $\quad a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \dfrac{2}{\pi}\int_0^{\pi} \sin x \cdot \cos nx\, dx$

$$= \frac{1}{\pi}\int_0^{\pi} [\sin(1+n)x + \sin(1-n)x]\, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi} \quad (n \neq 1)$$

$$= \frac{-1}{\pi}\left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n} - \frac{1}{1+n} - \frac{1}{1-n}\right]_0^{\pi} \quad (n \neq 1)$$

$$= \frac{-1}{\pi}\left[\frac{(-1)^{n+1}-1}{1+n} + \frac{(-1)^{n+1}-1}{1-n}\right]$$

$$= \frac{-1}{\pi}\left[(-1)^{n+1}\left\{\frac{1}{1+n} + \frac{1}{1-n}\right\} - \left\{\frac{1}{1+n} + \frac{1}{1-n}\right\}\right]$$

$$= \frac{-1}{\pi}\left[(-1)^{n+1} \cdot \frac{2}{1-n^2} - \frac{2}{1-n^2}\right]$$

$$= \frac{2}{\pi(n^2-1)}[(-1)^{n+1} - 1] = \frac{-2}{\pi(n^2-1)}[1 + (-1)^n] \quad (n \neq 1)$$

$$\therefore \quad a_n = \begin{cases} 0, & \text{if } n \text{ is odd and } n \neq 1 \\ \dfrac{-4}{\pi(n^2-1)}, & \text{if } n \text{ is even} \end{cases}$$

For $n = 1$, $a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cdot \cos x \, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$

$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi} = \frac{-1}{2\pi}(\cos 2\pi - 1) = 0$$

Substituting the values of a_0, a_1 and a_n in (1), we get

$$|\sin x| = \frac{2}{\pi} + \sum_{n=2,4,6,\ldots}^{\infty} \frac{-4}{\pi(n^2-1)} \cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6,\ldots}^{\infty} \frac{\cos nx}{n^2-1}$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2,4,6,\ldots}^{\infty} \frac{\cos nx}{n^2-1}$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1} \qquad \text{(Replacing } n \text{ by } 2n\text{)}$$

Hence, $|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \ldots\right)$

Ex. 9: Obtain Fourier series for the function $f(x)$ given by

$$f(x) = \begin{cases} -\frac{1}{2}(\pi + x), \text{ for } -\pi < x \le 0 \\ \frac{1}{2}(\pi - x), \text{ for } 0 \le x < \pi \end{cases}$$

Sol. Since $f(-x) = \frac{-1}{2}(\pi - x)$ in $(-\pi, 0) = -f(x)$ in $(0, \pi)$

and $f(-x) = \frac{1}{2}(\pi + x)$ in $(0, \pi) = -f(x)$ in $(-\pi, 0)$

$\therefore f(x)$ is an odd function in $(-\pi, \pi)$

Hence $a_0 = a_n = 0$. Now let

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}(\pi - x) \sin nx \, dx$

$$= \frac{1}{\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[(\pi - x)\left(\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{\pi},$$ since the first and second terms are vanishes at $x = \pi$.

$$\therefore b_n = \frac{1}{\pi}\left[(\pi - 0)\left(\frac{1}{n}\right) + 0\right] = \frac{1}{n}$$

Substituting the value of b_n in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots$$

which is the required Fourier series.

Ex. 10: (*a*) Is $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}, 0 \le x \le \pi \end{cases}$ **even?**

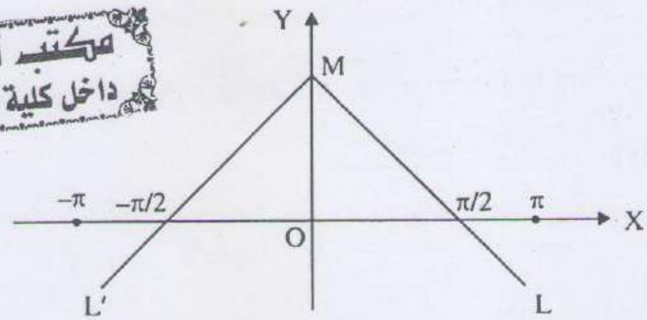
(*b*) If so, find the Fourier series for the function.

(*c*) Deduce that $\dfrac{1}{1^3} + \dfrac{1}{3^3} + \dfrac{1}{5^3} + \dfrac{1}{7^3} + \ldots = \dfrac{\pi^2}{8}$. **[JNTU 2003 (Set No. 4)]**

Sol. (*a*) Since $f(-x) = 1 - \dfrac{2x}{\pi}$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$

and $f(-x) = 1 + \dfrac{2x}{\pi}$ in $(0, \pi) = f(x)$ in $(-\pi, 0)$

$\therefore$ $f(x)$ is an even function in $(-\pi, \pi)$ and it is symmetrical about the y–axis.



(*b*) Since $f(x)$ is an even function in $(-\pi, \pi)$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(1)$$

where $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)dx$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left(x - \frac{x^2}{\pi}\right)_0^{\pi} = 0$$

and $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\cos nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}, \text{ by parts}$$

$$= \frac{2}{\pi}\left(\frac{-2\cos n\pi}{n^2\pi} + \frac{2}{n^2\pi}\right) \quad [\because \sin n\pi = 0]$$

$$= \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right] \qquad [\because \cos n\pi = (-1)^n]$$

Hence $\quad a_n = \frac{8}{n^2\pi^2}$, if n is odd

$= 0$, if n is even

$\therefore\ a_1 = \frac{8}{\pi^2},\ a_3 = \frac{8}{3^2\pi^2},\ a_5 = \frac{8}{5^2\pi^2}, \ldots$

and $a_2 = a_4 = a_6 = \ldots = 0$

Thus substituting the values of a's in (1), we get

$$f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \qquad \ldots(2)$$

as the required Fourier series.

(*c*) Putting $x = 0$ in (2), we get

$$f(0) = 1 = \frac{8}{\pi^2}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

which is the required result.

Ex.11. Prove that in the interval $-\pi < x < \pi$, sin*h* $ax = \frac{2}{\pi}\sinh a\pi \sum_{n=1}^{\infty} \frac{(-1)^n \cdot n\sin nx}{n^2 + a^2}$

Sol. Let $f(x) = \sinh ax = \frac{e^{ax} - e^{-ax}}{2}$

Now $f(-x) = \frac{e^{-ax} - e^{ax}}{2} = -\left(\frac{e^{ax} - e^{-ax}}{2}\right) = -\sinh ax = -f(x)$

So $f(x)$ is an odd function.

Let $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$

Then $\quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \sinh ax \sin nx\, dx$

$$= \frac{2}{\pi}\int_0^{\pi}\frac{e^{ax} - e^{-ax}}{2}\cdot \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} e^{ax}\sin nx\, dx - \int_0^{\pi} e^{-ax}\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right\}_0^{\pi} - \left\{\frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi) - \frac{1}{a^2+n^2}(-n)\right\} - \left\{\frac{e^{-a\pi}}{a^2+n^2}\right\}(-n\cos n\pi) - \frac{1}{a^2+n^2}(-n)\right]$$

$$= \frac{n}{\pi(a^2+n^2)}\left[-e^{a\pi}(-1)^n + 1 + (-1)^n e^{-a\pi} - 1\right] \quad [\because\ \cos n\pi = (-1)^n]$$

$$= \frac{n}{\pi(a^2+n^2)} \cdot (-1)^n \left[e^{-a\pi} - e^{a\pi} \right]$$

$$= \frac{n(-1)^{n+1}}{\pi(a^2+n^2)} (e^{a\pi} - e^{-a\pi}) = \frac{n(-1)^{n+1}}{\pi(a^2+n^2)} \cdot 2\sinh a\pi \qquad \ldots (2)$$

Substituting (2) in (1), we get

$$f(x) = \sum_{n=1}^{\infty} \frac{2n(-1)^{n+1}}{\pi(a^2+n^2)} \cdot \sinh a\pi \cdot \sin nx$$

i.e., $\sinh ax = \frac{2}{\pi} \sinh a\pi \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot n \cdot \sin nx}{n^2+a^2}$

Ex.12. If $f(x) = \cosh ax$, expand $f(x)$ as a Fourier series in $(-\pi, \pi)$.

Sol. Let $f(x) = \cosh ax = \frac{e^{ax} + e^{-ax}}{2}$

Then $f(-x) = \frac{e^{-ax} + e^{ax}}{2} = \cosh ax = f(x)$

$\therefore f(x)$ is an even function.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots (1)$

Then $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \frac{e^{ax} + e^{-ax}}{2} dx = \frac{1}{\pi} \int_0^{\pi} (e^{ax} + e^{-ax}) dx$

$$= \frac{1}{\pi} \left[\frac{e^{ax}}{a} + \frac{e^{-ax}}{-a} \right]_0^{\pi}$$

$$= \frac{1}{a\pi} (e^{a\pi} - e^{-a\pi}) = \frac{2}{a\pi} \left(\frac{e^{a\pi} - e^{-a\pi}}{2} \right) = \frac{2}{a\pi} \sinh a\pi$$

and $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$

$$= \frac{2}{\pi} \int_0^{\pi} \cosh ax \cos nx \, dx = \frac{1}{\pi} \int_0^{\pi} (e^{ax} + e^{-ax}) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} e^{ax} \cos nx \, dx + \int_0^{\pi} e^{-ax} \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\left\{ \frac{e^{ax}}{a^2+n^2} (a \cos nx + n \sin nx) \right\}_0^{\pi} + \left\{ \frac{e^{-ax}}{a^2+n^2} (-a \cos nx + n \sin nx) \right\}_0^{\pi} \right]$$

$$= \frac{1}{\pi} \left[\left\{ \frac{e^{a\pi}}{a^2+n^2} (a \cos n\pi + 0) - \frac{1}{a^2+n^2} (a+0) \right\} + \left\{ \frac{e^{-a\pi}}{a^2+n^2} (-a \cos n\pi + 0) - \frac{1}{a^2+n^2} (-a+0) \right\} \right]$$

$$= \frac{a}{\pi(a^2+n^2)}\left[\left\{e^{a\pi}(-1)^n - 1\right\} + \left\{-e^{-a\pi}(-1)^n + 1\right\}\right]$$

$$= \frac{a(-1)^n}{\pi(a^2+n^2)}(e^{a\pi} - e^{-a\pi}) = \frac{2a(-1)^n}{\pi(n^2+a^2)}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right)$$

$$= \frac{2a(-1)^n \sinh a\pi}{\pi(n^2+a^2)}$$

Substituting the values of a_0 and a_n in (1), we get

$$f(x) = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty} \frac{2a(-1)^n \sinh a\pi}{\pi(n^2+a^2)} \cdot \cos nx$$

$$= \frac{2a}{\pi} \cdot \sinh a\pi \left[\frac{1}{2a^2} + \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2+a^2}\right]$$

This is the required Fourier series.

EXERCISE 10 (B)

1. Expand the function $f(x) = x^3$ as a Fourier series in the interval $-\pi < x < \pi$.
2. Find the Fourier series to represent the function $f(x) = x\cos x, -\pi < x < \pi$
3. Find the Fourier series to represent the function $f(x) = |\cos x|, \ -\pi < x < \pi$
4. If x lies between $-\pi$ and π and a is neither zero nor an integer, prove that

   $$\cos ax = \frac{2a\sin a\pi}{\pi}\left(\frac{1}{2a^2} + \frac{\cos x}{1^2-a^2} - \frac{\cos 2x}{2^2-a^2} + \frac{\cos 3x}{3^2-a^2} - \dots\right)$$

   Hence prove that $\pi\cot a\pi = \frac{1}{a} + \frac{2a}{a^2-1} + \frac{2a}{a^2-2^2} - \frac{2a}{a^2-3^2} + \dots$
5. Find the Fourier series to represent the function $f(x) = \sqrt{1-\cos x}$ in $(-\pi, \pi)$
6. Find Fourier series of $\frac{\pi\cosh ax}{2\sinh a\pi}$ in $(-\pi, \pi)$
7. Obtain Fourier series for the function $f(x)$ given by :

   $$f(x) = \begin{cases} -x, \text{ for } -\pi < x \le 0 \\ x, \text{ for } 0 < x < \pi \end{cases}$$

   and deduce the value of $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$
8. A function $f(x)$ is defined as follows :

   $$f(x) = \begin{cases} -x^2 - \pi x & \text{if} -\pi \le x \le 0 \\ x^2 - \pi x & \text{if } 0 \le x \le \pi \end{cases}$$

   Show that $f(x) = \frac{-8}{\pi}\left(\frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \dots\right)$

9. Given : $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$

Is the function $f(x)$ even? If, so find the Fourier series for $f(x)$.

10. A periodic function $f(x)$ is defined as follows :

$f(x) = 2$, when $-\pi < x < 0$

$= 1$ when $0 \le x \le \pi$

and $f(x + 2\pi) = f(x)$. Find the Fourier series for $f(x)$.

ANSWERS

1. $x^3 = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}(6\pi - \pi^2 n^2)\sin nx$

2. $x\cos x = \frac{-1}{2}\sin x + 2\sum_{n=2}^{\infty} \frac{(-1)^n n}{n^2 - 1}\sin nx$

3. $|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left(\frac{1}{3}\cos 2x - \frac{1}{15}\cos 4x + \ldots\right)$

5. $f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \frac{\cos nx}{4n^2 - 1}$

6. $\frac{\pi \cosh ax}{2\sinh a\pi} = \frac{1}{2a} + \sum_{n=1}^{\infty} \frac{a(-1)^n}{a^2 + n^2}\cos nx$

7. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right); \frac{\pi^2}{8}$

9. $f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left(\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right)$

10. $f(x) = \frac{3}{2} - \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]$

10.8 HALF RANGE FOURIER SERIES

It is often required to obtain Fourier series of a function $f(x)$ in the interval $(0, \pi)$.

The Sine Series : If it be required to express $f(x)$ as a sine series in $(0, \pi)$, we define an odd function $f_1(x)$ in $(-\pi, \pi)$, identical with $f(x)$ in $(0, \pi)$. That is, we extend the function reflecting it w.r.t. the origin, so that $f(-x) = -f(x)$. Hence the half range sine series in $(0, \pi)$ is given by

$$\boxed{\begin{aligned} & f(x) = \sum_{n=1}^{\infty} b_n \sin nx \\ & \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx \end{aligned}}$$

The Cosine Series : If it be required to express $f(x)$ as a cosine series, we define an even function $f_2(x)$ in $(-\pi, \pi)$, identical with $f(x)$ in $(0, \pi)$. That is, we extend the function reflecting it with respect to the y-axis, so that $f(-x) = f(x)$.

Hence the half range cosine series in $(0, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$

and $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \; dx$

Note : (*i*) Suppose $f(x) = x$ in $[0, \pi]$. It can have Fourier Cosine series expansion as well as Fourier Sine series expansion in $[0, \pi]$.

(*ii*) Similarly $f(x) = x^2$ in $[0, \pi]$ can have Fourier Cosine series expansion as well as Fourier sine series expansion in $[0, \pi]$.

We note that, if $f(x)$ satisfies Dirichlet's conditions in $(0, \pi)$ we can have valid expansion of $f(x)$ in terms of sines only or cosines only in the interval.

EXAMPLES

Ex. 1: Find the half-range cosine and sine series for the function $f(x) = x$ in the range $0 \leq x \leq \pi$.

OR

Prove that the function $f(x) = x$ can be expanded in a series of cosines in $0 < x < \pi$ as $x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]$

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. **[JNTU 2003, 2003S (Set No. 1)]**

Sol. The Cosine Series.

The half range cosine series expansion of $f(x)$ in $[0, \pi]$ is given by

$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \ldots(1)$$

where $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \; dx$

Hence $a_0 = \frac{2}{\pi} \int_0^{\pi} x\, dx = \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi} = \pi$

and $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \; dx = \frac{2}{\pi} \int_0^{\pi} x \cos nx \; dx$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1 \cdot \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\cos nx}{n^2}\right)_0^{\pi} = \frac{2}{\pi n^2}\,[(-1)^n - 1]$$

$$\therefore \quad a_n = \begin{cases} 0, \text{ for } n \text{ even} \\ \dfrac{-4}{\pi n^2}, \text{ for } n \text{ odd} \end{cases}$$

Substituting these values in (1), we get

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,...} \frac{1}{n^2} \cos nx$$

or $$x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right) \qquad ...(2)$$

which is the required series for x.

Deduction.

When $x = 0, f(x) = 0$ i.e. $f(0) = 0$

Putting $x = 0$ in (2), we get

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$ **[JNTU 2003S]**

The Sine Series. The half range sine series expansion of $f(x)$ is given by **[JNTU 2003, 2005 (Set No.2)]**

$$f(x) = x = \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(3)$$

where $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x) \sin nx \, dx$

Hence $$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} = \frac{2}{\pi}\left(-\pi\frac{\cos n\pi}{n}\right) = (-1)^{n+1}\frac{2}{n}$$

Substituting the value of b_n in (3), we get

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots\right]$$

Ex. 2: Find the half range sine series for $f(x) = x(\pi - x)$, in $0 < x < \pi$.

Deduce that $\dfrac{1}{1^3} - \dfrac{1}{3^3} + \dfrac{1}{5^3} - \dfrac{1}{7^3} + \dots = \dfrac{\pi^3}{32}$ **[JNTU Dec 2002, 2003 (Set No. 2)]**

Sol. The Fourier sine series expansion of $f(x)$ in $(0, \pi)$ is

$$f(x) = x(\pi - x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

Hence $$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx \, dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2}{n^3}(1 - \cos n\pi)\right] = \frac{4}{\pi n^3}\left(1 - (-1)^n\right)$$

$$\therefore b_n = \begin{cases} 0, & \text{when } n \text{ is even} \\ \dfrac{8}{\pi n^3}, & \text{when } n \text{ is odd} \end{cases}$$

Hence $x(\pi - x) = \sum_{n = 1,3,5,\ldots} \frac{8}{\pi n^3} \sin nx$

or $x(\pi - x) = \frac{8}{\pi}\left(\sin x + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right)$ ...(1)

Which is the required Fourier sine series.

Deduction.

Putting $x = \pi/2$ in (1), we get

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{3\pi}{2} + \frac{1}{5^3}\sin\frac{5\pi}{2} + \ldots\right)$$

$$\Rightarrow \quad \frac{\pi^2}{4} = \frac{8}{\pi}\left[1 + \frac{1}{3^3}\sin\left(\pi + \frac{\pi}{2}\right) + \frac{1}{5^3}\sin\left(2\pi + \frac{\pi}{2}\right) + \frac{1}{7^3}\sin\left(3\pi + \frac{\pi}{2}\right) + \ldots\right]$$

or $\quad \frac{\pi^2}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$

Ex. 3: Express $f(x) = \sin x$ as Fourier cosine series in $(0, \pi)$.

Hence show that $\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} = \frac{1}{2}$

Sol. The half range Fourier cosine series is given by

$$f(x) = \sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Here $\quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x \, dx = \frac{2}{\pi}(-\cos x)_0^{\pi} = \frac{-2}{\pi}(-1 - 1) = \frac{4}{\pi}$$

and $\quad a_n = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx \, dx$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(1+n)x + \sin(1-n)x]\, dx = -\frac{1}{\pi}\left[\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\left(\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{1-n}\right) - \left(\frac{1}{1+n} + \frac{1}{1-n}\right)\right], \; n \neq 1$$

$$= -\frac{1}{\pi}\left[\left(\frac{1}{1+n}+\frac{1}{1-n}\right)(\cos\pi\cos n\pi - 1)\right]$$

$$= \frac{2}{\pi(n^2-1)}\left[-\cos n\pi - 1\right] = \frac{-2}{\pi(n^2-1)}[1+(-1)^n],\ n \neq 1$$

$$= \frac{-4}{\pi(n-1)(n+1)} \text{ or } 0$$

according as n is even or odd $(n \neq 1)$

Now $$a_1 = \frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$$

$$= \frac{1}{\pi}\left(\frac{-\cos 2x}{2}\right)_0^{\pi} = -\frac{1}{2\pi}(\cos 2\pi - \cos 0) = -\frac{1}{2\pi}(1-1) = 0$$

$\therefore\ a_1 = a_3 = a_5 = \ldots = 0$ and $a_2 = \dfrac{-4}{(2-1)(2+1)},\ a_4 = \dfrac{-4}{(4-1)(4+1)}$

Thus the half-range cosine series is

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4,6\ldots}\frac{\cos nx}{n^2-1} \qquad \ldots(1)$$

or $$\sin x = \frac{2}{\pi} - \frac{4\cos 2x}{\pi(2-1)(2+1)} - \frac{4\cos 4x}{\pi(4-1)(4+1)} - \frac{4\cos 6x}{\pi(6-1)(6+1)} - \ldots$$

$$= \frac{4}{\pi}\left(\frac{1}{2} - \frac{\cos 2x}{1\cdot 3} - \frac{\cos 4x}{3\cdot 5} - \frac{\cos 6x}{5\cdot 7} - \ldots\right)$$

Which is the required series for $\sin x$. It is easy to see that (1) can be written as

$$\sin x = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2nx}{4n^2-1} \qquad \ldots(2)$$

Deduction.

Putting $x = 0$ in (2), we obtain

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} \text{ or } \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}$$

which is the required result.

Ex. 4: Obtain the half-range sine series for e^x in $(0, \pi)$.

Sol. The half range sine series expansion of e^x in $(0, \pi)$ is given by

$$e^x = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{\pi}\int_0^{\pi} e^x \sin nx\,dx$$

i.e. $$b_n = \frac{2}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{\pi} \quad \left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

or $\quad b_n = \frac{2}{\pi}\left\{\left[\frac{e^{\pi}}{1+n^2}(0-n\cos n\pi)\right]-\frac{1}{1+n^2}(0-n)\right\}$

$$= \frac{2}{\pi}\left[\frac{(-1)^{n+1} n e^{\pi}}{1+n^2}+\frac{n}{1+n^2}\right] = \frac{2n}{\pi(1+n^2)}\left[1+(-1)^{n+1}e^{\pi}\right]$$

Thus the half-range sine series for e^x is

$$e^x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n\left[1+(-1)^{n+1}e^{\pi}\right]}{n^2+1}\sin nx$$

$$= \frac{2}{\pi}\left[\frac{1+e^{\pi}}{1^2+1}\sin x+\frac{2(1-e^{\pi})}{2^2+1}\sin 2x+\frac{3(1+e^{\pi})}{3^2+1}\sin 3x+\ldots\right]$$

Ex. 5: Obtain the Fourier cosine series for $f(x) = x\sin x$, $0 < x < \pi$ and show that

$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\ldots = \frac{\pi-2}{4}$ **[JNTU 2002, 2006 (Set No. 1)]**

Sol. Let $f(x) = x\sin x = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx$ ...(1)

where $\quad a_0 = \frac{2}{\pi}\int_0^{\pi}f(x)dx = \frac{2}{\pi}\int_0^{\pi}x\sin x\,dx$

$$= \frac{2}{\pi}\left[x(-\cos x)+(\sin x)\right]_0^{\pi} = \frac{2}{\pi}\left[-\pi\cos\pi+\sin\pi\right] = \frac{2}{\pi}(\pi) = 2$$

and $\quad a_n = \frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}x\sin x\cos nx\,dx$

$$= \frac{1}{\pi}\int_0^{\pi}x\left[\sin(n+1)x-\sin(n-1)x\right]dx \quad (n\neq 1)$$

$$= \frac{1}{\pi}\left\{x\left[\frac{-\cos(n+1)x}{n+1}+\frac{\cos(n-1)x}{n-1}\right]-(1)\left[\frac{-\sin(n+1)x}{(n+1)^2}+\frac{\sin(n-1)x}{(n-1)^2}\right]\right\}_0^{\pi} \quad (n\neq 1)$$

$$= \frac{-1}{(n+1)}\cos(n+1)\pi+\frac{1}{n-1}\cos(n-1)\pi$$

$$= \frac{(-1)^{n-1}}{n-1}-\frac{(-1)^{n+1}}{n+1} = \frac{2(-1)^{n+1}}{n^2-1},\ (n\neq 1)$$

$\therefore\ a_2 = \frac{-2}{1.3};\quad a_3 = \frac{2}{2.4};\ a_4 = \frac{-2}{3.5};\ a_5 = \frac{2}{4.6};\ \ldots$

Now $\quad a_1 = \frac{2}{\pi}\int_0^{\pi}x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^{\pi}x.\sin 2x\,dx$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right)-\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi} = \frac{1}{\pi}\left[\frac{-\pi}{2}\cos 2\pi\right] = \frac{-1}{2}$$

From (1), we have

$$x\sin x = 1-\frac{1}{2}\cos x-\frac{2}{1.3}\cos 2x+\frac{2}{2.4}\cos 3x-\frac{2}{3.5}\cos 4x+\ldots \qquad \ldots(2)$$

Deduction :

Putting $x=\frac{\pi}{2}$ in (2), we obtain

$$\frac{\pi}{2}=1+\frac{2}{1.3}-\frac{2}{3.5}+\frac{2}{5.7}-\frac{2}{7.9}+\ldots \text{ or } 1+\frac{2}{1.3}-\frac{2}{3.5}+\frac{2}{5.7}-\frac{2}{7.9}+\ldots=\frac{\pi}{2}$$

or $$\frac{2}{1.3}-\frac{2}{3.5}+\frac{2}{5.7}-\frac{2}{7.9}+\ldots=\frac{\pi}{2}-1=\frac{\pi-2}{2}$$

or $$\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\frac{1}{7.9}+\ldots=\frac{\pi-2}{4}$$

Ex. 6: Find the half-range sine series for the function $f(x)=\dfrac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}}$ in $(0,\pi)$

Sol. Let $f(x)=\sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots(1)$

Then $$b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx=\frac{2}{\pi}\int_0^{\pi}\frac{e^{ax}-e^{-ax}}{e^{a\pi}-e^{-a\pi}}\cdot\sin nx\,dx$$

$$=\frac{2}{\pi(e^{a\pi}-e^{-a\pi})}\left[\int_0^{\pi} e^{ax}\sin nx\,dx-\int_0^{\pi} e^{-ax}\sin nx\,dx\right]$$

$$=\frac{2}{\pi(e^{a\pi}-e^{-a\pi})}\left[\left\{\frac{e^{ax}}{a^2+n^2}(a\sin nx-n\cos nx)\right\}_0^{\pi}-\left\{\frac{e^{-ax}}{a^2+n^2}(-a\sin nx-n\cos nx)\right\}_0^{\pi}\right]$$

$$=\frac{2}{\pi(e^{a\pi}-e^{-a\pi})}\left[-\frac{e^{a\pi}}{a^2+n^2}\cdot n(-1)^n+\frac{n}{a^2+n^2}+\frac{e^{-a\pi}}{a^2+n^2}\cdot n(-1)^n-\frac{n}{a^2+n^2}\right]$$

$$=\frac{2n(-1)^n}{\pi(e^{a\pi}-e^{-a\pi})}\left[\frac{e^{-a\pi}-e^{a\pi}}{n^2+a^2}\right]=\frac{2n(-1)^{n+1}}{\pi(n^2+a^2)} \qquad \ldots(2)$$

Substituting (2) in (1), we get

$$f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{a^2+n^2}\sin nx$$

$$=\frac{2}{\pi}\left[\frac{\sin x}{a^2+1^2}-\frac{2\sin 2x}{a^2+2^2}+\frac{3\sin 3x}{a^2+3^2}-\cdots\right]$$

Ex. 7 If $f(x)=\begin{cases}1 \text{ in } 0<x<\dfrac{\pi}{2}\\ -1 \text{ in } \dfrac{\pi}{2}<x<\pi\end{cases}$ expand $f(x)$ in a series of cosines.

Sol. Let $$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx \qquad \ldots(1)$$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\left[\int_0^{\pi/2} 1\cdot dx + \int_{\pi/2}^{\pi}(-1)dx\right]$$

$$= \frac{2}{\pi}\left[(x)_0^{\pi/2} - (x)_{\pi/2}^{\pi}\right] = \frac{2}{\pi}\left[\left(\frac{\pi}{2}-0\right)-\left(\pi-\frac{\pi}{2}\right)\right] = 0$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\left[\int_0^{\pi/2} 1\cdot\cos nx\, dx + \int_{\pi/2}^{\pi}(-1)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\pi/2} - \left(\frac{\sin nx}{n}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{n}\left(\sin\frac{n\pi}{2}-0\right)-\frac{1}{n}\left(0-\sin\frac{n\pi}{2}\right)\right]$$

$$= \frac{2}{\pi}\cdot\frac{2}{n}\sin\frac{n\pi}{2} = \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

$\therefore$ $a_n = 0$ when n is even i.e., $a_2 = a_4 = a_6 = \ldots = 0$

Hence $$f(x) = \sum_{n=1,3,5,\ldots}^{\infty}\frac{4}{n\pi}\sin\frac{n\pi}{2}\cos nx$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi}{2}\cos x + \frac{1}{3}\sin\frac{3\pi}{2}\cos 3x + \frac{1}{5}\sin\frac{5\pi}{2}\cos 5x + \cdots\right)$$

$$= \frac{4}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots\right)$$

This is the required Fourier series.

Ex.8. Represent the function $f(x) = \begin{cases} x \text{ for } 0 < x < \frac{\pi}{2} \\ \frac{\pi}{2} \text{ for } \frac{\pi}{2} < x < \pi \end{cases}$ **by Fourier sine series.**

Sol. Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ ... (1)

Then $$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\frac{\pi}{2}\sin nx\, dx + \int_{\pi/2}^{\pi}(\pi - x)\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\left(-\frac{\cos nx}{n}\right)_0^{\pi/2} + \left\{(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\frac{\pi}{2n}\left(\cos\frac{n\pi}{2}-1\right) - \left\{-\frac{\pi}{2n}\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{n\pi}{2}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

i.e. $$b_1 = \frac{2}{\pi}\left[\frac{\pi}{2}+\sin\frac{\pi}{2}\right]=\frac{2}{\pi}\left(\frac{\pi}{2}+1\right)=1+\frac{2}{\pi}$$

$$b_2 = \frac{2}{\pi}\left[\frac{\pi}{4}+\frac{1}{4}\sin\pi\right]=\frac{2}{\pi}\left(\frac{\pi}{4}+0\right)=\frac{1}{2}$$

$$b_3 = \frac{2}{\pi}\left[\frac{\pi}{6}+\frac{1}{9}\sin\frac{3\pi}{2}\right]=\frac{2}{\pi}\left(\frac{\pi}{6}-\frac{1}{9}\right)=\frac{1}{3}-\frac{2}{9\pi}$$

$$b_4 = \frac{2}{\pi}\left[\frac{\pi}{8}+\frac{1}{16}\sin 2\pi\right]=\frac{2}{\pi}\left(\frac{\pi}{8}+0\right)=\frac{1}{4}$$

and so on.

Substituting the values of b's in (1), we get

$$f(x) = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \ldots$$
$$= \left(1+\frac{2}{\pi}\right)\sin x+\frac{1}{2}\sin 2x+\left(\frac{1}{3}-\frac{2}{9\pi}\right)\sin 3x+\frac{1}{4}\sin 4x+\cdots$$

Ex.9. Find the Fourier cosine series of $f(x)=\begin{cases}\cos x \text{ when } 0<x<\dfrac{\pi}{2}\\ 0 \text{ when } \dfrac{\pi}{2}<x<\pi\end{cases}$

Sol. Let $$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \qquad \ldots (1)$$

Then $$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx=\frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\,dx+\int_{\pi/2}^{\pi}0.\,dx\right]$$

$$= \frac{2}{\pi}(\sin x)_0^{\pi/2}=\frac{2}{\pi}(1-0)=\frac{2}{\pi} \qquad \ldots (2)$$

and $$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cdot\cos nx\,dx+\int_{\pi/2}^{\pi}0\cdot\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi/2}2\cos nx\cos x\,dx \qquad \ldots (3)$$

$$= \frac{1}{\pi}\int_0^{\pi/2}[\cos(n+1)x+\cos(n-1)x]dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} \quad (n\neq 1)$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right] \ (n \neq 1)$$

$$= \frac{1}{\pi}\left[\frac{\cos\frac{n\pi}{2}\cdot\sin\frac{\pi}{2}}{n+1}-\frac{\cos\frac{n\pi}{2}\cdot\sin\frac{\pi}{2}}{n-1}\right] \ \left(\because \cos\frac{\pi}{2}=0\right)$$

$$\therefore \quad a_n = \frac{\cos\frac{n\pi}{2}}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}\right] \quad (n \neq 1)$$

$$= \frac{-2\cos\frac{n\pi}{2}}{\pi(n^2-1)} \ (n \neq 1) \qquad \ldots (4)$$

If $n = 1$, then

$$a_1 = \frac{1}{\pi}\int_0^{\pi/2} 2\cos^2 x \, dx \ \text{[From (3)]}$$

$$= \frac{1}{\pi}\int_0^{\pi/2} (1+\cos 2x)dx = \frac{1}{\pi}\left(x+\frac{\sin 2x}{2}\right)_0^{\pi/2}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right] = \frac{1}{2} \qquad \ldots (5)$$

Substituting (2), (4) and (5) in (1), we get

$$f(x) = \frac{a_0}{2}+a_1\cos x+\sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{1}{\pi}+\frac{1}{2}\cos x-\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{\cos\frac{n\pi}{2}}{n^2-1}\cos nx$$

$$= \frac{1}{\pi}+\frac{1}{2}\cos x-\frac{2}{\pi}\left[\frac{\cos\pi}{(2-1)(2+1)}\cos 2x+\frac{\cos\frac{3\pi}{2}}{(3-1)(3+1)}\cos 3x+\frac{\cos 4\pi}{(4-1)(4+1)}\cos 4x+\cdots\right]$$

$$= \frac{1}{\pi}+\frac{1}{2}\cos x-\frac{2}{\pi}\left[-\frac{\cos 2x}{1.3}+\frac{\cos 4x}{3.5}-\frac{\cos 6x}{5.7}+\cdots\right]$$

$$= \frac{1}{\pi}+\frac{1}{2}\cos x+\frac{2}{\pi}\left[\frac{\cos 2x}{1.3}-\frac{\cos 4x}{3.5}+\frac{\cos 6x}{5.7}-\cdots\right]$$

Ex. 10: If $f(x) = x$, **for** $0 < x < \frac{\pi}{2}$

$$= \pi - x, \text{ for } \frac{\pi}{2} < x < \pi$$

Show that (*i*) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right]$

(*ii*) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right]$

Sol. (*i*) Sine Series. The Sine series expansion of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{.... (1)}$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx$

Hence $b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx \, dx\right]$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right) - 1\cdot\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{-\cos nx}{n}\right) + \left(\frac{-\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\left[\frac{-\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) - 0\right] + \left[0 + \frac{\pi}{2n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]\right\}$$

$$= \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$$

$\therefore \quad b_n = 0$ when n is even

i.e. $\quad b_2 = b_4 = b_6 = \ldots = 0$

and $\quad b_n = \frac{4}{\pi n^2}\sin\frac{n\pi}{2}$, when n is odd.

Hence $\quad f(x) = \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2}\sin\frac{n\pi}{2}\sin nx$

$$= \frac{4}{\pi}\left[\frac{\sin\frac{\pi}{2}}{1^2}\sin x + \frac{\sin\frac{3\pi}{2}}{3^2}\sin 3x + \frac{\sin\frac{5\pi}{2}}{5^2}\sin 5x + \ldots\right]$$

$$= \frac{4}{\pi}\left[\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \ldots\right]$$

(ii) Cosine Series. The Cosine series expansion of $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \qquad \text{.... (2)}$$

where $\quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$

Hence $\quad a_0 = \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)dx\right] = \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \left(\pi x - \frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{2\pi^2}{8}\right) = \frac{\pi}{2}$$

and $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos nx\, dx\right]$$

$$= \frac{2}{\pi}\left\{\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{\sin nx}{n}\right) + 1\cdot\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\right) + \left(-\frac{\cos n\pi}{n^2} - \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n^2} - \frac{\cos n\pi}{n^2}\right]$$

$\therefore \quad a_n = \frac{2}{\pi n^2}\left(2\cos\frac{n\pi}{2} - 1 - (-1)^n\right)$

when n is odd, $a_n = 0$

i.e., $a_1 = a_3 = a_5 = \ldots = 0$

and $a_n = \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - 2\right]$ when n is even $\qquad$ .... (3)

Putting $n = 2, 4, 6, 8, 10, \ldots$ in (3), we get

$$a_2 = \frac{2}{\pi}\left(\frac{2\cos\pi - 2}{2^2}\right) = \frac{2}{\pi}(-1) = \frac{-2}{\pi}$$

$$a_4 = \frac{2}{\pi}\left(\frac{2\cos 2\pi - 2}{4^2}\right) = 0$$

$$a_6 = \frac{2}{\pi}\left(\frac{2\cos 2\pi - 2}{6^2}\right) = \frac{2}{\pi}\left(\frac{-4}{6^2}\right) = \frac{-2}{\pi 3^2}$$

$$a_8 = \frac{2}{\pi}\left(\frac{2\cos 4\pi - 2}{8^2}\right) = 0$$

$$a_{10} = \frac{2}{\pi}\left(\frac{2\cos 5\pi - 2}{10^2}\right) = \frac{2}{\pi}\left(\frac{-4}{10^2}\right) = \frac{-2}{\pi 5^2}$$

Substituting the values of a_0 and a_n in (2), we get

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[(-1)\cos 2x - \frac{1}{3^2}\cos 6x - \frac{1}{5^2}\cos 10x - \ldots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{1}{1^2}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{5^2}\cos 10x + \ldots\right)$$

Aliter :

Since $a_n = 0$ when n is odd, we have

$$f(x) = \frac{\pi}{4} + \sum_{n=2,4,6,\ldots}^{\infty} \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]\cos nx$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{1}{n^2}\frac{2}{\pi(4n^2)}[2\cos n\pi - 1 - (-1)^{2n}]\cos 2nx \quad \text{(Replacing } n \text{ with } 2n\text{)}$$

$$= \frac{\pi}{4} + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}[2(-1)^n - 2]\cos 2nx = \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^n - 1]}{n^2}\cos 2nx$$

$$= \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{-2}{n^2}\right) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos 2nx$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{12}\cos 2x + \frac{1}{3^2}\cos 6x + \frac{1}{52}\cos 10x + \ldots\right]$$

Ex. 11: Expand $f(x) = \cos x$, $0 < x < \pi$ in half range sine series

[JNTU 2005(Set No. 4)]

Sol. Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ ... (1)

Then $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi}\cos x \sin nx\, dx$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin nx\cos x\, dx = \frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x + \sin(n-1)x]dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \quad (n \neq 1)$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-(-1)^{n+1}}{n+1}+\frac{(-1)^n}{n-1}+\frac{1}{n+1}+\frac{1}{n-1}\right] \quad (n \neq 1)$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n(-1)^2}{n+1}+\frac{(-1)^n}{n-1}+\frac{1}{n+1}+\frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left\{\frac{1}{n+1}+\frac{1}{n-1}\right\}+\left\{\frac{1}{n+1}+\frac{1}{n-1}\right\}\right]$$

$$= \frac{1}{\pi}\left[\{(-1)^n+1\}\left(\frac{1}{n+1}+\frac{1}{n-1}\right)\right] \quad (n \neq 1)$$

$$= \frac{2n}{\pi}\left[\frac{1+(-1)^n}{n^2-1}\right] \quad (n \neq 1)$$

$\therefore \quad b_n = 0$ when n is odd and $n \neq 1$

$$= \frac{4^n}{\pi(n^2-1)} \text{ when } n \text{ is even}$$

If $n = 1$, then

$$b_1 = \frac{2}{\pi}\int_0^\pi \cos x \sin x \, dx = \frac{1}{\pi}\int_0^\pi \sin 2x \, dx$$

$$= \frac{1}{\pi}\left(-\frac{\cos 2x}{2}\right)_0^\pi = \frac{-1}{2\pi}\,(\cos 2\pi - \cos 0) = \frac{-1}{2\pi}(1-1) = 0$$

Thus $b_n = 0$ when n is odd

$$= \frac{4n}{\pi(n^2-1)} \text{ when } n \text{ is even}$$

Substituting the values of b's in (1), we get

$$f(x) = \sum_{n=2,4,6,\ldots}^{\infty} \frac{4n}{\pi(n^2-1)}\sin nx = \frac{4}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{n}{n^2-1}\sin nx$$

i.e. $\cos x = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2n}{4n^2-1}\sin 2nx$ ($\because$ n is even, replace n by $2n$)

$$= \frac{8}{\pi}\left(\frac{1}{3}\sin 2x + \frac{2}{15}\sin 4x + \frac{3}{35}\sin 6x + \cdots\right)$$

EXERCISE 10 (C)

1. Express $f(x) = 1$ as Fourier sine series in $(0, \pi)$.
2. Obtain cosine and sinc series for $f(x) = \pi - x$ in $[0, \pi]$.
3. Obtain cosine and sine series for the function $f(x) = x^2$ in $[0, \pi]$.

   and hence find sum of the series $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

   [JNTU Dec. 2002, 2003S, 2003, 2004 (Set No. 4)]

4. Express $f(x) = x^3$ as Fourier sine series in $(0, \pi)$.

5. Find the Fourier sine series of e^{ax} in $(0, \pi)$.

6. Obtain the half-range sine series for the function $f(x) = \dfrac{\pi x}{8}(\pi - x)$ in the range $0 \le x \le \pi$.

7. Find the half-range cosine series for the function $f(x) = x \sin x$ in $(0, \pi)$. Hence deduce that **[JNTU 2000]**

$$\frac{1}{2}+\frac{1}{1\cdot 3}-\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}-\dots = \frac{\pi}{4} \text{ or } 1+\frac{2}{1.3}-\frac{2}{3.5}+\frac{2}{5.7}-\dots = \frac{\pi}{2}$$

8. Obtain the Fourier expansion of $x \cos x$ as a sine series in the interval $0 < x < \pi$.

9. Express $f(x) = 1 + 2\cos x + 3\cos^2 x + 4\cos^3 x$ as Fourier cosine series in $(0, \pi)$.

10. Express $f(x) = 2 \sin 2x \cos x$ as Fourier sine series in $(0, \pi)$.

11. If
$$\begin{aligned} f(x) &= \frac{\pi x}{4}, \text{ for } 0 < x < \frac{\pi}{2} \\ &= \frac{\pi}{4}(\pi - x), \text{ for } \frac{\pi}{2} < x < \pi \end{aligned}$$

Show that (*i*) $f(x) = \sin x - \dfrac{1}{3^2}\sin 3x + \dfrac{1}{5^2}\sin 5x - \dfrac{1}{7^2}\sin 7x + \dots$

(*ii*) $f(x) = \dfrac{\pi^2}{16} - \dfrac{1}{2}\left(\dfrac{1}{1^2}\cos x + \dfrac{1}{3^2}\cos 6x + \dots\right)$

12. Obtain the Fourier sine and cosine series in $0 < x < \pi$ for the function

$$f(x) = \begin{cases} \pi/3, & \text{when } 0 < x < \pi/3 \\ 0, & \text{when } \pi/3 < x < 2\pi/3 \\ -\pi/3, & \text{when } 2\pi/3 < x < \pi \end{cases}$$

ANSWERS

1. $1 = \dfrac{4}{\pi}\left[\sin x + \dfrac{\sin 3x}{3} + \dfrac{\sin 5x}{5} + \dots\right]$

2. $\pi - x = \dfrac{\pi}{2} + \dfrac{4}{\pi}\left[\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \dots\right]$

$\pi - x = 2\left(\sin x + \dfrac{1}{2}\sin 2x + \dfrac{1}{3}\sin 3x + \dots\right)$

3. $x^2 = \dfrac{\pi^2}{3} - 4\left(\dfrac{\cos x}{1^2} - \dfrac{1}{2^2}\cos 2x + \dfrac{1}{3^2}\cos 3x - \dfrac{1}{4^2}\cos 4x + \dots\right);$

$$x^2 = \frac{2}{\pi}\left[(\pi^2 - 4)\sin x - \frac{1}{2}\pi^2 \sin 2x + \frac{1}{3}\left(\pi^2 - \frac{4}{3^2}\right)\sin 3x - \frac{1}{4}\pi^2 \sin 4x + \frac{1}{5}\left(\pi^2 - \frac{4}{5^2}\right)\sin 5x - \dots\right]$$

4. $f(x) = 2\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{6}{n^3} - \frac{\pi^2}{n}\right)\sin nx$

5. $e^{ax} = \dfrac{2}{\pi}\left[\dfrac{1+e^{a\pi}}{a^2+1^2}\sin x + \dfrac{2(1-e^{a\pi})}{a^2+2^2}\sin 2x + \dfrac{3(1+e^{a\pi})}{a^2+3^2}\sin 3x + \dots\right]$

6. $f(x) = \frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots$

7. $x \sin x = 1 - \frac{\cos x}{2} - \frac{2\cos 2x}{1 \cdot 3} + \frac{2\cos 3x}{2 \cdot 4} - \frac{2\cos 4x}{3 \cdot 5} + \ldots$

8. $x \cos x = -\frac{1}{2}\sin x + 2\left[\frac{2\sin 2x}{1 \cdot 3} - \frac{3\sin 3x}{2 \cdot 4} + \frac{4\sin 4x}{3 \cdot 5} - \ldots\right]$

9. $f(x) = \frac{5}{2} + 5\cos x + \frac{3}{2}\cos 2x + \cos 3x$

10. $2\sin 2x \cos x = \sin x + \sin 3x$

12. $f(x) = \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{4}\sin 8x + \frac{1}{5}\sin 10x + \frac{1}{7}\sin 4x + \ldots$

$f(x) = \frac{2}{\sqrt{3}}\left[\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x - \frac{1}{11}\cos 11x + \ldots\right]$

10.9 INTERVALS OTHER THAN $(-\pi, \pi)$ AND $(0, 2\pi)$

So far we have considered the intervals $(-\pi, \pi)$ and $(0, 2\pi)$. In many engineering problems, the period of the function to be expanded is not 2π but some other quantity say $2l$. In order to apply earlier discussions to functions of period $2l$, this interval must be converted to the length 2π.

10.10 FOURIER SERIES OF $f(x)$ DEFINED IN $[C, C + 2l]$

It can be seen that role played by the functions

$1, \cos x, \cos 2x, \cos 3x \ldots, \sin x, \sin 2x, \ldots$

in expanding a function $f(x)$ defined in $[C, C + 2\pi]$ as a Fourier series, will be played by

$1, \cos\left(\frac{\pi x}{l}\right), \cos\left(\frac{2\pi x}{l}\right), \cos\left(\frac{3\pi x}{l}\right), \ldots, \sin\left(\frac{\pi x}{l}\right), \sin\left(\frac{2\pi x}{l}\right), \sin\left(\frac{3\pi x}{l}\right) \ldots$

in expanding a function $f(x)$ defined in $[C, C + 2l]$.

It can be verified directly that, when m, n are integers

$$\int_C^{C+2l} \sin\left(\frac{m\pi x}{l}\right) \cdot \cos\left(\frac{n\pi x}{l}\right) dx = 0$$

$$\int_C^{C+2l} \sin\left(\frac{m\pi x}{l}\right) \cdot \sin\left(\frac{n\pi x}{l}\right) = \begin{cases} 0, \text{ if } m \neq n \\ l, \text{ if } m = n \neq 0 \\ 0, \text{ if } m = n = 0 \end{cases}$$

$$\int_C^{C+2l} \cos\left(\frac{m\pi x}{l}\right) \cdot \cos\left(\frac{n\pi x}{l}\right) dx = \begin{cases} 0, \text{ if } m \neq n \\ l, \text{ if } m = n \neq 0 \\ 2l, \text{ if } m = n = 0 \end{cases}$$

10.11 FOURIER SERIES OF $f(x)$ DEFINED IN $[0, 2l]$:

Let $f(x)$ be defined in $[0, 2l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as (or is given by)

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right] \qquad \ldots(1)$$

where $\qquad a_n = \dfrac{1}{l}\displaystyle\int_0^{2l} f(x)\cos\dfrac{n\pi x}{l}\,dx \qquad$ ...(2)

and $\qquad b_n = \dfrac{1}{l}\displaystyle\int_0^{2l} f(x)\sin\dfrac{n\pi x}{l}\,dx. \qquad$ ...(3)

10.12 FOURIER SERIES OF $f(x)$ DEFINED IN $[-l, l]$

Let $f(x)$ be defined in $[-l, l]$ and be periodic with period $2l$. Its Fourier series expansion is defined as (or is given by)

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$$

[(same as in (1))]

where $\qquad a_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\cos\dfrac{n\pi x}{l}\,dx$

and $\qquad b_n = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\sin\dfrac{n\pi x}{l}\,dx$

10.13 FOURIER SERIES FOR EVEN AND ODD FUNCTIONS IN $[-l, l]$

Let $f(x)$ be defined in $[-l, l]$. If $f(x)$ is even, $f(x)\cos\dfrac{n\pi x}{l}$ is also even

$$\therefore\ a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx;$$

and $f(x)\sin\dfrac{n\pi x}{l}$ is odd

$$\therefore\ b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0, \text{ for all } n.$$

Hence if $f(x)$ is defined in $[-l, l]$ and is even, its Fourier series expansion is given by

$$\boxed{\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l} \\ \text{where}\quad a_n &= \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx \end{aligned}}$$

Similarly if $f(x)$ is defined in $[-l, l]$ and is odd, its Fourier series expansion is given by

$$\boxed{\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{l} \\ \text{where}\ b_n &= \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\,dx \end{aligned}}$$

Once again, here we remarks that the even nature or odd nature of the function is to be considered only when we deal with the interval $[-l, l]$.

Note : In the above discussion if we put $2l = 2\pi$ (*i.e.,*) $l = \pi$, we get the discussion regarding the intervals $[0, 2\pi]$ and $[-\pi, \pi]$ as special cases.

10.14 FOURIER SERIES OF $f(x)$ DEFINED IN $[c, c + 2l]$

This will be same as in (1) where a_n, b_n are given as in (2), (3) with the limits of integration 0, $2l$ replaced by c, $c + 2l$.

EXAMPLES

Ex. 1: Express $f(x) = x^2$ as a Fourier series in $[-l, l]$ **[JNTU 2002]**

Sol. Since $f(-x) = (-x)^2 = x^2 = f(x)$, therefore $f(x)$ is an even function

Hence the Fourier series of $f(x)$ in $[-l, l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad ...(1)$$

where $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

Hence $a_0 = \frac{2}{l}\int_0^l x^2 dx = \frac{2}{l}\left(\frac{x^3}{3}\right)_0^l = \frac{2l^2}{3}$

Also $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\left[x^2\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right) - 2x\left(\frac{-\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right) + 2\left(\frac{-\sin\frac{n\pi x}{l}}{n^3\pi^3/l^3}\right)\right]_0^l$$

$= \frac{2}{l}\left[2x\cdot\frac{\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right]_0^l$, since the first and last terms vanish at both upper and lower limits

$$\therefore a_n = \frac{2}{l}\left[2l\,\frac{\cos n\pi}{n^2\pi^2/l^2}\right] = \frac{4l^2\cos n\pi}{n^2\pi^2} = \frac{(-1)^n 4l^2}{n^2\pi^2}$$

Substituting these values in (1), we get

$$x^2 = \frac{l^2}{3} + \sum_{n=1}^{\infty}\frac{(-1)^n 4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{\cos(\pi x/l)}{1^2} - \frac{\cos(2\pi x/l)}{2^2} + \frac{\cos(3\pi x/l)}{3^2} - \frac{\cos(4\pi x/l)}{4^2} + \ldots\right]$$

Ex. 2: Obtain Fourier series expansion for $\sin ax$ in the interval. $-l < x < l$.

Sol. Since $\sin ax$ is an odd function in $(-l, l)$, therefore the required series is of the form

$$\sin ax = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad ...(1)$$

Then $b_n = \frac{2}{l}\int_0^l \sin ax \sin\frac{n\pi x}{l}\,dx$

$$= \frac{1}{l}\int_0^l \left[\cos\left(a - \frac{n\pi}{l}\right)x - \cos\left(a + \frac{n\pi}{l}\right)x\right]dx$$

$$= \frac{1}{l}\left[\frac{\sin\left(a - \frac{n\pi}{l}\right)x}{a - \frac{n\pi}{l}} - \frac{\sin\left(a + \frac{n\pi}{l}\right)x}{a + \frac{n\pi}{l}}\right]_0^l = \frac{1}{l}\left[\frac{\sin(al - n\pi)}{a - \frac{n\pi}{l}} - \frac{\sin(al + n\pi)}{a + \frac{n\pi}{l}}\right]$$

$$= \frac{1}{l}\left[\frac{\sin al \cos n\pi}{a - \frac{n\pi}{l}} - \frac{\sin al \cos n\pi}{a + \frac{n\pi}{l}}\right] \quad [\because \sin n\pi = 0]$$

$$= \frac{(-1)^n \sin al}{l}\left[\frac{2n\pi / l}{a^2 - \frac{n^2\pi^2}{l^2}}\right]$$

$$\therefore \; b_n = \frac{(-1)^{n+1}\sin al}{n^2\pi^2 - a^2l^2}$$

Substituting these values in (1), we get

$$\sin ax = \sin al \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2 - a^2l^2}\sin\frac{n\pi x}{l}$$

$$= \sin al\left[\frac{1}{\pi^2 - a^2l^2}\sin\frac{\pi x}{l} - \frac{1}{2^2\pi^2 - a^2l^2}\sin\frac{2\pi x}{l} + \frac{1}{3^2\pi^2 - a^2l^2}\sin\frac{3\pi x}{l} - \ldots\right]$$

This is the required Fourier series expansion.

Note. Compare this working with the case of the interval $[-\pi, \pi]$.

Ex. 3: Find the Fourier series to represent $1 - x^2$ in the interval $-1 \leq x \leq 1$.

Sol. Since $1 - x^2$ is an even function, therefore the required series is of the form

$$1 - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where $l = 1$

$$\therefore \; 1 - x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \qquad \ldots(1)$$

where $a_0 = \frac{2}{1}\int_0^1 (1 - x^2)\,dx = 2\left(x - \frac{x^3}{3}\right)_0^1 = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$

and $a_n = \frac{2}{1}\int_0^1 (1-x^2)\cos n\pi x \, dx \quad (\because l = 1)$

$$= 2\left[(1-x^2)\frac{\sin n\pi x}{n\pi} - (-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= \frac{-4\cos n\pi}{n^2\pi^2} = \frac{(-1)^{n+1}4}{n^2\pi^2}$$

Substituting these values in (1), we get

$$1 - x^2 = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}}{n^2}\right)\cos n\pi x$$

$$= \frac{2}{3} + \frac{4}{\pi^2}\left[\cos\pi x - \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x - \ldots\right]$$

which is the required Fourier series

Ex. 4: Develop $f(x)$ as Fourier series in $(-2, 2)$, if

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

Sol. The Fourier series of $f(x)$ in $(-l, l)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \ldots(1)$$

Here $l = 2$.

Hence (1) becomes

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \quad \ldots(2)$$

Then $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0\cdot dx + \int_{-1}^{1} k\cdot dx + \int_{1}^{2} 0\cdot dx\right] = \frac{1}{2}\int_{-1}^{1} k\,dx$$

$$= \frac{k}{2}\int_{-1}^{1} dx = \frac{k}{2}(x)_{-1}^{1} = k$$

and $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx \quad (\because l = 2)$

$$= \frac{1}{2}\left[\int_{-2}^{-1} 0\cdot\cos\frac{n\pi x}{2}dx + \int_{-1}^{1} k\cdot\cos\frac{n\pi x}{2}dx + \int_{1}^{2} 0\cdot\cos\frac{n\pi x}{2}dx\right]$$

$$= \frac{k}{2} \int_{-1}^{1} \cos \frac{n\pi x}{2} dx$$

$$= k \int_{0}^{1} \cos \frac{n\pi x}{2} dx, \text{ since } \cos \frac{n\pi x}{2} \text{ is even function}$$

$$= k \left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} \right) = \frac{2k}{n\pi} \cdot \sin \frac{n\pi}{2}$$

$\therefore \ a_n = 0$ when n is even

i.e. $\quad a_2 = a_4 = a_6 = \ldots = 0$

and $a_1 = \frac{2k}{\pi}, \ a_3 = \frac{-2k}{3\pi}, \ a_5 = \frac{2k}{5\pi}, \ldots$

Finally $\quad b_n = \frac{1}{2} \int_{-2}^{2} f(x) \sin \frac{n\pi x}{2} dx = \frac{1}{2} \int_{-1}^{1} k . \sin \frac{n\pi x}{2} dx$

$$= \frac{k}{2}(0) = 0, \text{ since } \sin \frac{n\pi x}{2} \text{ is odd}$$

Substituting the values of a's and b's in (2), we get

$$f(x) = \frac{k}{2} + \frac{2k}{\pi} \left[\cos \frac{\pi x}{2} - \frac{1}{3} \cos \frac{3\pi x}{2} + \frac{1}{5} \cos \frac{5\pi x}{2} - \frac{1}{7} \cos \frac{7\pi x}{2} + \ldots \right]$$

which is the required Fourier series.

Ex. 5: Find the Fourier series of $f(x) = \frac{\pi - x}{2}$ in $0 < x < 2$

Sol. Here length of interval is $2l = 2 \ \therefore \ l = 1$

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right]$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos n\pi x + b_n \sin n\pi x \right], \text{since } l = 1 \qquad \ldots(1)$$

Then $\quad a_0 = \frac{1}{1} \int_0^2 f(x)\, dx = \int_0^2 \frac{\pi - x}{2}\, dx = \frac{1}{2} \left(\pi x - \frac{x^2}{2} \right)_0^2 = \pi - 1$

and $\quad a_n = \frac{1}{1} \int_0^2 f(x) \cos \frac{n\pi x}{1} dx \qquad (\because l = 1)$

$$= \int_0^2 \frac{\pi - x}{2} \cos n\pi x \, dx = \frac{1}{2} \int_0^2 (\pi - x) \cos n\pi x \, dx$$

$$= \frac{1}{2} \left[(\pi - x) \left(\frac{\sin n\pi x}{n\pi} \right) + \left(\frac{-\cos n\pi x}{n^2 \pi^2} \right) \right]_0^2 = \frac{1}{2} \left[\frac{-\cos 2n\pi}{n^2\pi^2} + \frac{1}{n^2\pi^2} \right]$$

$$= \frac{1}{2n^2\pi^2} (1 - \cos 2n\pi) = \frac{1}{2n^2\pi^2} (1 - 1) = 0$$

Finally $b_n = \frac{1}{1}\int_0^2 f(x)\sin\frac{n\pi x}{1}\,dx \qquad (\because l=1)$

$$= \frac{1}{2}\int_0^2 (\pi - x)\sin n\pi x\,dx$$

$$= \frac{1}{2}\left[(\pi-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - \frac{\sin n\pi x}{n^2\pi^2}\right]_0^2 = \frac{-1}{2n\pi}\left[(\pi-2)\cos 2n\pi - \pi\right]$$

$$= \frac{-1}{2n\pi}\left[\pi - 2 - \pi\right] = \frac{1}{n\pi}$$

Substituting the values of a's and b's in (1), we get

$$f(x) = \frac{\pi - x}{2} = \frac{\pi - 1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}.\sin n\pi x$$

which is the required Fourier series.

Ex. 6: Find a Fourier series with period 3 to represent $f(x) = x + x^2$ in (0, 3)

Sol. Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right)$ ...(1)

Here $2l = 3 \therefore l = 3/2$

Hence (1) becomes

$$f(x) = x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{3} + b_n\sin\frac{2n\pi x}{3}\right) \qquad ...(2)$$

where $a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{2}{3}.\int_0^3 (x+x^2)\,dx = \frac{2}{3}\left(\frac{x^2}{2}+\frac{x^3}{3}\right)_0^3 = 9$

and $a_n = \frac{1}{l}\int_0^2 f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{3}\int_0^3 (x+x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$

Integrating by parts, we obtain

$$a_n = \frac{2}{3}\left[\frac{3}{4n^2\pi^2} - \frac{9}{4n^2\pi^2}\right] = \frac{2}{3}\left(\frac{54}{4n^2\pi^2}\right) = \frac{9}{n^2\pi^2}$$

Finally $b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{3}\int_0^3 (x+x^2)\sin\left(\frac{2n\pi x}{3}\right)dx = \frac{-12}{n\pi}$$

Substituting the values of $a's$ and $b's$ in (2), we get

$$x + x^2 = \frac{9}{2} + \frac{9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) - \frac{12}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{2n\pi x}{3}\right)$$

Ex. 7: Expand $f(x) = e^{-x}$ as a Fourier series in the interval $(-1, 1)$.

Sol. Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

Here $l = 1$.

$$\therefore \quad e^{-x} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \dots (1)$$

Then $$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx = \int_{-1}^{1} e^{-x}dx \quad (\because l = 1)$$

$$= -\left(e^{-x}\right)_{-1}^{1} = -(e^{-1} - e) = e - e^{-1}$$

$$= 2\left(\frac{e - e^{-1}}{2}\right) = 2\sinh 1$$

and $$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \int_{-1}^{1} e^{-x}\cos n\pi x\, dx \quad (\because l = 1)$$

$$= \left[\frac{e^{-x}}{1+n^2\pi^2}(-\cos n\pi x + n\pi \sin n\pi x)\right]_{-1}^{1}$$

$$= \frac{1}{1+n^2\pi^2}\left[e^{-1}(-\cos n\pi + n\pi\sin n\pi) - e(-\cos n\pi - n\pi \sin n\pi)\right]$$

$$= \frac{1}{1+n^2\pi^2}\left[-e^{-1}(-1)^n + e(-1)^n\right] \quad (\because \sin n\pi = 0)$$

$$= \frac{(-1)^n}{1+n^2\pi^2}(e - e^{-1}) = \frac{(-1)^n 2\sin h1}{1+n^2\pi^2}$$

Finally $$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$$

$$= \int_{-1}^{1} e^{-x}\sin n\pi x\, dx \quad (\because l = 1)$$

$$= \left[\frac{e^{-x}}{1+n^2\pi^2}(-\sin n\pi x - n\pi\cos n\pi x)\right]_{-1}^{1}$$

$$= \frac{1}{1+n^2\pi^2}[e^{-1}(-\sin n\pi - n\pi\cos n\pi) - e(\sin n\pi - n\pi\cos n\pi)]$$

$$= \frac{1}{1+n^2\pi^2}[e^{-1}(0 - n\pi\cos n\pi) - e(0 - n\pi\cos n\pi)]$$

$$= \frac{1}{1+n^2\pi^2}[n\pi\cos n\pi(e - e^{-1})]$$

$$= \frac{n\pi(-1)^n}{1+n^2\pi^2} \cdot 2\left(\frac{e-e^{-1}}{2}\right)$$

$$= \frac{2n\pi(-1)^n}{1+n^2\pi^2} \sinh 1.$$

Substituting the values of a_0, a_n and b_n in (1), we get

$$f(x) = \sinh 1 + \sum_{n=1}^{\infty} \frac{2(-1)^n \sinh 1}{1+n^2\pi^2} \cos n\pi x + \sum_{n=1}^{\infty} \frac{2n\pi(-1)^n \sinh 1}{1+n^2\pi^2} \sin n\pi x$$

or $$e^{-x} = \sinh 1\left[1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2} \cos n\pi x + 2\sum_{n=1}^{\infty} \frac{(-1)^n n\pi}{1+n^2\pi^2} \sin n\pi x\right]$$

Ex. 8: Expand $f(x) = 3x^2 - 2$ as a Fourier series in the interval $(-3, 3)$.

Sol. Since $f(x) = 3x^2 - 2$ is an even function,

$\therefore \quad f(x) = \frac{a_0}{2} \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$

Here $l = 3$

Hence the required series is of the form

$$f(x) = 3x^2 - 2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{3} \qquad \ldots (1)$$

Then $$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{3}\int_0^3 (3x^2 - 2)dx$$

$$= \frac{2}{3}(x^3 - 2x)_0^3 = \frac{2}{3}(27-6) = 14 \qquad \ldots (2)$$

and $$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{3}\int_0^3 (3x^2-2)\cos\frac{n\pi x}{3}dx$$

$$= \frac{2}{3}\left[(3x^2-2)\cdot\frac{\sin\frac{n\pi x}{3}}{n\pi/3} - 6x\left(-\frac{\cos\frac{n\pi x}{3}}{n^2\pi^2/9}\right) + 6\left(-\frac{\sin\frac{n\pi x}{3}}{n^3\pi^3/27}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\frac{18\times 9}{n^2\pi^2}\cos n\pi\right] = \frac{108(-1)^n}{n^2\pi^2} \qquad \ldots (3)$$

Substituting (2) and (3) in (1), we get

$$3x^2 - 2 = 7 + \sum_{n=1}^{\infty} \frac{108(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{3} = 7 - \frac{108}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{3}$$

$$= 7 - \frac{108}{\pi^2}\left[\cos\frac{\pi x}{3} - \frac{1}{4}\cos\frac{2\pi x}{3} + \frac{1}{9}\cos\pi x - \cdots\right]$$

which is the required Fourier series.

Ex. 9: Express $f(x) = 1 + \sin x$ as a Fourier series in the interval (–1, 1).

Sol. Here $l = 1$

$\therefore$ The required series is of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x \qquad \ldots (1)$$

$$\text{Then } a_0 = \frac{1}{l}\int_{-1}^{1} f(x)dx = \int_{-1}^{1}(1+\sin x)dx \;\; (\because l = 1)$$

$$= \int_{-1}^{1} dx + \int_{-1}^{1} \sin x \, dx$$

$$= (x)_{-1}^{1} + 0 \;\; (\because \sin x \text{ is odd function})$$

$$= 1 + 1 = 2$$

and

$$a_n = \frac{1}{l}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{l}dx = \int_{-1}^{1}(1+\sin x)\cdot\cos n\pi x \, dx \;\; (\because l = 1)$$

$$= \int_{-1}^{1}\cos n\pi x \, dx + \int_{-1}^{1}\sin x\cdot\cos n\pi x \, dx$$

$$= 2\int_{0}^{1}\cos n\pi x \, dx + 0$$

($\because$ $\cos n\pi x$ is even and $\sin x.\cos n\pi x$ is odd function)

$$= 2\left(\frac{\sin n\pi x}{n\pi}\right)_0^1 = \frac{2}{n\pi}(\sin n\pi - 0) = \frac{2}{n\pi}(0-0) = 0$$

$$\text{Finally } b_n = \frac{1}{l}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{l}dx$$

$$= \int_{-1}^{1}(1+\sin x)\sin n\pi x \, dx \;\; (\because l = 1)$$

$$= \int_{-1}^{1}\sin n\pi x \, dx + \int_{-1}^{1}\sin x\cdot\sin n\pi x \, dx$$

$$= 0 + 2\int_{0}^{1}\sin x\cdot\sin n\pi x \, dx$$

$$= \int_{0}^{1} 2\sin x\cdot\sin n\pi x \, dx$$

$$= \int_{0}^{1}[\cos(n\pi-1)x - \cos(n\pi+1)x]dx$$

$$= \left[\frac{\sin(n\pi-1)x}{n\pi-1} - \frac{\sin(n\pi+1)x}{n\pi+1}\right]_0^1$$

$$= \frac{\sin(n\pi-1)}{n\pi-1} - \frac{\sin(n\pi+1)}{n\pi+1}$$

$$= \frac{1}{n\pi-1}[\sin n\pi \cdot \cos 1 - \cos n\pi \cdot \sin 1] - \frac{1}{n\pi+1}[\sin n\pi \cdot \cos 1 + \cos n\pi \cdot \sin 1]$$

$$= \frac{1}{n\pi+1}(0 - \cos n\pi \cdot \sin 1) - \frac{1}{n\pi+1}(0 + \cos n\pi \cdot \sin 1)$$

$$\therefore \quad b_n = -\cos n\pi \cdot \sin 1\left(\frac{1}{n\pi-1} + \frac{1}{n\pi+1}\right)$$

$$= \frac{2n\pi(-1)^{n+1}\sin 1}{n^2\pi^2 - 1}$$

Substituting the values of a's and b's in (1), we get

$$1 + \sin x = 1 + 2\pi \sin 1 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2 - 1} \sin n\pi x$$

Ex. 10: If $f(x) = |x|$, expand $f(x)$ as a Fourier series in the interval $(-2, 2)$.

Sol. Here $l = 2$

Since $|x|$ is an even function,

$\therefore$ The required series is of the form

$$|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \quad \dots (1)$$

where $$a_0 = \frac{2}{l}\int_0^l f(x)dx = \int_0^2 |x|\,dx \quad (\because l = 2)$$

$$= \int_0^2 x\,dx = \left(\frac{x^2}{2}\right)_0^2 = \frac{1}{2}(4-0) = 2 \quad \dots (2)$$

and $$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 |x|\cos\frac{n\pi x}{2}dx \quad (\because l = 2)$$

$$= \int_0^2 x\cos\frac{n\pi x}{2}dx \quad (\because 0 < x < 2)$$

$$= \left[x \cdot \frac{\sin\frac{n\pi x}{2}}{n\pi/2} - 1 \cdot \left(-\frac{\cos\frac{n\pi x}{2}}{n^2\pi^2/4}\right)\right]_0^2$$

$$= \left(0 + \frac{\cos n\pi}{n^2\pi^2/4}\right) - \left(0 + \frac{1}{n^2\pi^2/4}\right)$$

$$= \frac{(-1)^n - 1}{n^2\pi^2/4} = \frac{4}{n^2\pi^2}\,[(-1)^n - 1]$$

$$\therefore \quad \left.\begin{aligned} a_n &= 0 \text{ when } n \text{ is even} \\ &= \frac{-8}{n^2\pi^2} \text{ when } n \text{ is odd} \end{aligned}\right\} \qquad \text{... (3)}$$

Substituting (2) and (3) in (1), we get

$$|x| = 1 - \frac{8}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{2} = 1 - \frac{8}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \cdots\right]$$

Ex. 11: Find the Fourier series expansion for $f(x)$, if $f(x) = \begin{cases} 2 \text{ if } -2 \le x \le 0 \\ x \text{ if } 0 < x < 2 \end{cases}$

[JNTU 2006 (Set No.2)]

Sol. Here $l = 2$

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \quad (\because l = 2)$$

Then $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx\right] = \frac{1}{2}\left[2(x)_{-2}^{0} + \left(\frac{x^2}{2}\right)_0^2\right] = 3$$

and $$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx \quad (\because l = 2)$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\cdot\cos\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[2\cdot\left(\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\right)_{-2}^{0} + \left\{x\left(\frac{\sin\frac{n\pi x}{2}}{n\pi/2}\right) - 1\cdot\left(-\frac{\cos\frac{n\pi x}{2}}{n^2\pi^2/4}\right)\right\}_0^2\right]$$

$$= \frac{1}{2}\left[\frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right] = \frac{2}{n^2\pi^2}\,[(-1)^n - 1]$$

$$\therefore \quad \begin{aligned} a_n &= 0 \text{ when } n \text{ is even} \\ &= -\frac{4}{n^2\pi^2} \text{ when } n \text{ is odd} \end{aligned}$$

Finally $$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx = \frac{1}{2}\left[\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[2\left(-\frac{\cos\frac{n\pi x}{2}}{n\pi/2}\right)_{-2}^{0} + \left\{x\left(-\frac{\cos\frac{n\pi x}{2}}{n\pi/2}\right) - 1\left(-\frac{\sin\frac{n\pi x}{2}}{n^2\pi^2/4}\right)\right\}_0^2\right]$$

$$= \frac{1}{2}\left[-\frac{4}{n\pi} + \frac{4}{n\pi}\cos n\pi\right] + \frac{1}{2}\left[-\frac{4}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi\right] = \frac{1}{2}\left(\frac{-4}{n\pi}\right) = \frac{-2}{n\pi}$$

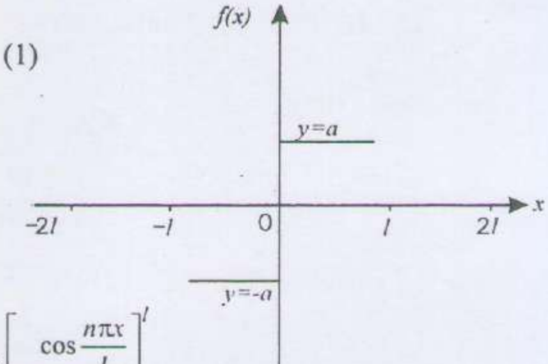
Substituting the values of a_0 and a_n in (1), we get

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2} \sum_{n=1,3,5,...}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin\frac{n\pi x}{2}$$

Ex. 12: Find the Fourier series of the function $f(x) = \begin{cases} -a \text{ when } -l < x < 0 \\ a \text{ when } 0 < x < l \end{cases}$

Sol. Since $f(x)$ is an odd function,

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \text{... (1)}$$



where $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l a\sin\frac{n\pi x}{l}\,dx = \frac{2a}{l}\left[-\frac{\cos\frac{n\pi x}{l}}{n\pi/l}\right]_0^l$$

$$= \frac{-2a}{n\pi}\left(\cos\frac{n\pi x}{l}\right)_0^l = \frac{-2a}{n\pi}(\cos n\pi - \cos 0)$$

$$= \frac{-2a}{n\pi}[(-1)^n - 1] = \frac{2a}{n\pi}[1-(-1)^n]$$

$\therefore \quad b_n = 0$ of n is even

$= \dfrac{4a}{n\pi}$ if n is odd

Substituting the values of b's in (1), we get

$$f(x) = \sum_{n=1,3,5,...}^{\infty} \frac{4a}{n\pi}\sin\frac{n\pi x}{l} = \frac{4a}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right)$$

Ex. 13: Find the Fourier series of the function $f(x) = \begin{cases} \frac{1}{2} + x \text{ when } & -1 \le x \le 0 \\ \frac{1}{2} - x \text{ when } & 0 \le x \le 1 \end{cases}$

Sol. Since $f(-x) = \frac{1}{2} - x$ in $(-1, 0) = f(x)$ in $(0, 1)$

and $\quad f(-x) = \frac{1}{2} + x$ in $(0, 1) = f(x)$ in $(-1, 0)$

$\therefore$ $f(x)$ is an even function.

Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x \quad (\because l = 1)$

Then $a_0 = \frac{2}{l}\int_0^l f(x)dx = 2\int_0^1 f(x)dx = 2\int_0^1 \left(\frac{1}{2} - x\right)dx$

$$= 2\left(\frac{x}{2} - \frac{x^2}{2}\right)_0^1 = \left(x - x^2\right)_0^1 = (1-1) - (0-0) = 0$$

and $a_n = \frac{2}{l}\int_0^1 f(x)\cos\frac{n\pi x}{l}dx = 2\int_0^1 f(x)\cos n\pi x\, dx$

$$= 2\int_0^1\left(\frac{1}{2} - x\right)\cos n\pi x\, dx$$

$$= 2\left[\left(\frac{1}{2} - x\right)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[\left(0 - \frac{\cos n\pi}{n^2\pi^2}\right) - \left(0 - \frac{1}{n^2\pi^2}\right)\right]$$

$$= \frac{2}{n^2\pi^2}(1 - \cos n\pi) = \frac{2}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$\therefore$ $a_n = 0$ if n is even

$= \frac{4}{n^2\pi^2}$ if n is odd

Substituting the values of a_0 and a_n in (1), we get

$$f(x) = \frac{4}{\pi^2}\sum_{n=1,3,5...}^{\infty}\frac{1}{n^2}\cos n\pi x = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos(2n-1)\pi x \quad (\because n \text{ is odd})$$

10.15 HALF RANGE EXPANSION OF $f(x)$ in $[0, l]$

Sometimes we will be interested in finding the expansion of $f(x)$ defined in $[0, l]$ in terms of sines only or in terms of cosines only. Suppose we want the expansion of $f(x)$ in terms of sine series only. Define $f_1(x) = f(x)$ in $[0, l]$ and $f_1(-x) = -f_1(x)$ for all x with $f_1(2l + x) = f_1(x)$. Then $f_1(x)$ is an odd function in $[-l, l]$. Hence its Fourier series expansion is given by

$$f_1(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}dx \qquad ...(1)$$

where

$$b_n = \frac{2}{l}\int_0^l f_1(x)\, dx \qquad ...(2)$$

The above expansion is valid for x in $[-l, l]$. In particular for x in $[0, l]$, $f_1(x) = f(x)$ and hence

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \, dx \qquad ...(3)$$

where $$b_n = \frac{2}{l} \int_0 f(x) \sin \frac{n\pi x}{l} dx \qquad ....(4)$$

This expansion in (3) is called the half-range sine series expansion of $f(x)$ in $[0, l]$.

If we want the half-range expansion of $f(x)$ in $[0, l]$, only in terms of cosines, define $f_1(x) = f(x)$ in $[0, l]$ and $f_1(-x) = f_1(x)$ for all x with $f_1(x + 2l) = f_1(x)$.

Then $f_1(x)$ is even in $[-l, l]$ and hence its Fourier series expansion is given by

$$f_1(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \qquad ...(5)$$

where $$a_n = \frac{2}{l} \int_0^l f_1(x) \cos \frac{n\pi x}{l} dx \qquad ...(6)$$

The expansion is valid in $[-l, l]$ and hence in particular in $[0, l]$. But in $[0, l]$, $f_1(x) = f(x)$. Hence in $[0, l]$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \qquad ...(7)$$

where $$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx \qquad ...(8)$$

The expansion in (7) is called the half-range cosine series expansion of $f(x)$ in $[0, l]$.

Summary :

1. The (half range) sine series expansion of $f(x)$ in $[0, l]$ is given by

$$\boxed{\begin{array}{c} f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \\ \text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx. \end{array}}$$

2. The (half-range) cosine series expansion of $f(x)$ in $[0, l]$ is given by

$$\boxed{\begin{array}{c} f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \\ \text{where } b_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx. \end{array}}$$

EXAMPLES

Ex. 1: Find the half-range sine series of $f(x) = 1$ in $[0, l]$.

[JNTU 2004S (Set No. 3)]

Sol. The Fourier sine series of $f(x)$ in $[0, l]$ is given by

$$f(x) = 1 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

Here $\quad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l 1\cdot\sin\frac{n\pi x}{l}\,dx$

$$= \frac{2}{l}\left(\frac{-\cos\frac{n\pi x}{l}}{n\pi/l}\right)_0^l = \frac{2}{n\pi}\left(-\cos\frac{n\pi x}{l}\right)_0^l = \frac{2}{n\pi}(-\cos n\pi + 1) = \frac{2}{n\pi}\left[(-1)^{n+1}+1\right]$$

$\therefore \quad b_n = 0$, when n is even

$\quad = \frac{4}{n\pi}$, when n is odd

Hence the required Fourier series is

$$f(x) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4}{n\pi}\sin\frac{n\pi x}{l}$$

i.e., $\quad 1 = \frac{4}{\pi}\left(\sin\frac{n\pi}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \ldots\right)$

Ex.2. Obtain the half-range cosine and sine series for $f(x) = x$ in the interval $(0, l)$.

Sol. Cosine series

Let $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \quad \ldots (1)$

Then $\quad a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left(\frac{x^2}{2}\right)_0^l = \frac{1}{l}(l^2 - 0) = l$

and $\quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}dx$

$$= \frac{2}{l}\left[x\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right) - 1\cdot\left(-\frac{\cos\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 - \frac{l^2}{n^2\pi^2}\cos n\pi\right\} - \left\{0 + \frac{l^2}{n^2\pi^2}\right\}\right] \quad (\because \sin n\pi = 0)$$

$$= \frac{2}{l}\left[\frac{l^2}{n^2\pi^2}\{(-1)^n - 1\}\right] \quad [\because \cos n\pi = (-1)^n]$$

$\therefore \quad a_n =$, 0 if n is even

$\quad = -\frac{4l}{n^2\pi^2}$, if n is odd

Substituting the values of a's in (1), we get

$$x = \frac{l}{2} - \frac{4l}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{l}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \cdots\right)$$

Sine Series

Let $$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots (2)$$

Then $$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx = \frac{2}{l}\int_0^l x\sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[x\left(-\frac{\cos\frac{n\pi x}{l}}{n\pi/l}\right) - 1\cdot\left(-\frac{\sin\frac{n\pi x}{l}}{n^2\pi^2/l^2}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{-\frac{l^2}{n\pi}\cos n\pi + 0\right\} - \{0+0\}\right]$$

$$= -\frac{2l}{n\pi}(-1)^n = \frac{2l}{n\pi}(-1)^{n+1} \qquad \ldots (3)$$

Substituting (3) in (2), we get

$$x = \frac{2l}{\pi}\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}\sin\frac{n\pi x}{l}$$

$$= \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \cdots\right).$$

Ex. 3: Find the half-range cosine series expansion of $f(x) = x$ in [0, 2].

Sol. The half range Fourier cosine series is given by

$$f(x) = x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \qquad \ldots(1)$$

Here $l = 2$

Now $$a_0 = \frac{2}{l}\int_0^l f(x)\,dx = \frac{2}{2}\int_0^2 x\,dx = \left(\frac{x^2}{2}\right)_0^2 = 2$$

and $$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}dx$$

$$= \left[x\frac{\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)} + \frac{\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2 = \frac{\cos n\pi - 1}{n^2\pi^2/4} = \frac{4\left[(-1)^n - 1\right]}{n^2\pi^2} \quad (n \neq 1)$$

$\therefore \; a_n = 0$ when n is even

$= \dfrac{-8}{n^2\pi^2}$ if n is odd

Substituting the values of a_0 and a_n in (1), we get

$$x = 1 - \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{2}$$

$$= 1 - \frac{8}{\pi^2}\left[\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right]$$

which is the required Fourier series.

Ex. 4: Find the half-range sine series expansion of $f(x) = x^2$ in [0, 4].

Sol. The half-range Fourier sine series is given by

$$f(x) = x^2 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots(1)$$

where $b_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

Here $\quad l = 4$

$$\therefore \; b_n = \frac{2}{4}\int_0^4 x^2 \sin\frac{n\pi x}{4}\,dx$$

$$= \frac{1}{2}\left[x^2\left(\frac{-\cos\frac{n\pi x}{4}}{\frac{n\pi}{4}}\right) - 2x\left(\frac{-\sin\frac{n\pi x}{4}}{n^2\pi^2/16}\right) + 2\left(\frac{\cos\frac{n\pi x}{4}}{\frac{n^3\pi^3}{64}}\right)\right]_0^4$$

$$= \frac{1}{2}\left[\frac{-4}{n\pi}(16)\cos n\pi + \frac{32}{n^2\pi^2}(4)\sin n\pi + \frac{2\cdot 64}{n^3\pi^3}\cos n\pi - \frac{128}{n^3\pi^3}\right]$$

$$= \frac{1}{2}\left[\frac{(-1)^{n+1}64}{n\pi} + \frac{(-1)^n 128}{n^3\pi^3} - \frac{128}{n^3\pi^3}\right] = 32\left[\frac{2\left\{(-1)^n - 1\right\}}{n^3\pi^3} + \frac{(-1)^{n+1}}{n\pi}\right]$$

or $\qquad b_n = \dfrac{-32}{n\pi}$, when n is even

$= 32\left(\dfrac{-4}{n^3\pi^3} + \dfrac{1}{n\pi}\right)$ when n is odd.

Hence the required Fourier series is

$$x^2 = \sum_{n=2,4,6,\ldots}^{\infty} \frac{-32}{n\pi}\sin\frac{n\pi x}{4} + 32\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{-4}{n^3\pi^3} + \frac{1}{n\pi}\right)\frac{\sin n\pi x}{4}$$

Ex. 5: Find the half-range cosine series expansion of
$f(x) = x - x^2$ in $0 < x < 1$

Sol. The required series is of the form $x - x^2 = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos n\pi x$

where $a_n = 2\int_0^1 (x - x^2)\cos n\pi x \, dx \quad (\because l = 1)$

$$= 2\left[(x - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1 - 2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[\frac{-\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$= \frac{-2}{n^2\pi^2}(1 + \cos n\pi) = \frac{-2[1 + (-1)^n]}{n^2\pi^2}$$

$\therefore \quad a_n = 0$, when n is odd

$= \frac{-4}{n^2\pi^2}$, when n is even

Also, $a_0 = 2\int_0^1 (x - x^2)\, dx = 2\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}$

Hence $x - x^2 = \frac{1}{6} - \frac{4}{\pi^2}\left[\frac{1}{2^2}\cos 2\pi x + \frac{1}{4^2}\cos 4\pi x + \frac{1}{6^2}\cos 6\pi x + \ldots\right]$

which is the required series.

Ex. 6: Find half-range Fourier sine series for $f(x) = ax + b$, in $0 < x < 1$

[JNTU Dec. 2002, 2003, 2005 (Set No. 1), 2006S (Set No.2)]

Sol. The half-range Fourier sine series is given by

$$f(x) = ax + b = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad (\because l = 1)$$

where $b_n = \frac{2}{l}\int_0^l (ax + b)\sin\left(\frac{n\pi x}{l}\right)dx$

$$= 2\int_0^1 (ax + b)\sin n\pi x \, dx \quad (\because l = 1)$$

$$= 2\left\{(ax + b)\left(\frac{-1}{n\pi}\cos n\pi x\right) + \frac{a}{n^2\pi^2}\sin n\pi x\right\}_0^1$$

$$= 2\left[\frac{-1}{n\pi}(a + b)\cos n\pi + \frac{a}{n^2\pi^2}\sin n\pi + \frac{b}{n\pi}\right]$$

$$= \frac{2}{n\pi}\left[(-1)^{n+1}(a + b) + b\right]$$

$\therefore b_n = -\frac{2a}{n\pi}$, when n is even

$= \frac{2}{n\pi}(a + 2b)$, when n is odd

Hence the required Fourier series is

$$ax+b = \frac{2}{\pi}(a+2b)\sin \pi x - \frac{2a}{2\pi}\sin 2\pi x + \frac{2}{3\pi}(a+2b)\sin 3\pi x - \frac{2a}{4\pi}\sin 4\pi x + \ldots\ldots$$

Ex. 7: Find the half-range cosine series for $f(x) = x(2-x)$, in $0 \le x \le 2$ and hence find sum of the series

$$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

[JNTU 2002 (Set No. 2), 2003, 2004 (Set No. 3)]

Sol. The required series is of the form

$$x(2-x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \quad (\because l=2) \qquad \ldots(1)$$

where $\quad a_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$

$$=\frac{2}{2}\int_0^2 x(2-x)\cos\frac{n\pi x}{2}\,dx \quad (\because l=2)$$

$$=\int_0^2 (2x-x^2)\cos\frac{n\pi x}{2}\,dx$$

$$=\left[(2x-x^2)\frac{2}{n\pi}\left(\sin\frac{n\pi x}{2}\right)+(2-2x)\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}+(2)\frac{8}{n^3\pi^3}\sin\frac{n\pi x}{2}\right]_0^2, \text{ by parts}$$

$$=\frac{-8}{n^2\pi^2}\cos n\pi - \frac{8}{n^2\pi^2}=\frac{-8}{n^2\pi^2}\left[1+(-1)^n\right]$$

$\therefore \quad a_n = \dfrac{-16}{n^2\pi^2}$, when n is even

$= 0$, when n is odd

and $\quad a_0 = \dfrac{2}{2}\displaystyle\int_0^2 \left(2x-x^2\right)dx = \frac{4}{3}$

Substituting the values of a_0 and a_n in (1), we get

$$2x-x^2=\frac{2}{3}-\frac{16}{\pi^2}\sum_{n=2,4,6\ldots}^{\infty}\left(\frac{1}{n^2}\cos\frac{n\pi x}{2}\right)$$

$$=\frac{2}{3}-\frac{16}{\pi^2}\left(\frac{1}{2^2}\cos\pi x+\frac{1}{4^2}\cos 2\pi x+\frac{1}{6^2}\cos 3\pi x+\ldots\right)$$

$$=\frac{2}{3}-\frac{16}{\pi^2}\cdot\frac{1}{2^2}\left[\cos\pi x+\frac{1}{2^2}\cos 2\pi z+\frac{1}{3^2}\cos 3\pi x+\ldots\right]$$

$$\therefore\ 2x-x^2 = \frac{2}{3}-\frac{4}{\pi^2}\left(\cos\pi x+\frac{1}{2^2}\cos 2\pi x+\frac{1}{3^2}\cos 3\pi x+\ldots\right) \qquad \ldots(2)$$

Deduction :

Putting $x = 1$ in (2), we get

$$2-1=\frac{2}{3}-\frac{4}{\pi^2}\left(\cos\pi+\frac{1}{2}\cos 2\pi+\frac{1}{3^2}\cos 3\pi+\frac{1}{4^2}\cos 4\pi+...\right)$$

$$\Rightarrow \quad 1-\frac{2}{3}=\frac{-4}{\pi^2}\left(-1+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-...\right)$$

$$\Rightarrow \quad \frac{1}{3}=\frac{4}{\pi^2}\left(1-\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-...\right)$$

or $$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...=\frac{\pi^2}{12}$$

Ex. 8: Find the half-range cosine series expansion of $f(x)=\sin\left(\frac{\pi x}{l}\right)$ in the range $0<x<l$. **[JNTU 2004S, 2006S (Set No. 3)]**

Sol. The half-range Fourier Cosine series is given by

$$f(x)=\sin\left(\frac{\pi x}{l}\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{l}$$

where $$a_0=\frac{2}{l}\int_0^l f(x)\,dx=\frac{2}{l}\int_0^l \sin\frac{\pi x}{l}\,dx$$

$$=\frac{2}{l}\left[\frac{-\cos(\pi x/l)}{\pi/l}\right]_0^l=\frac{-2}{\pi}(\cos\pi-1)=\frac{4}{\pi}$$

and $$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx=\frac{1}{l}\int_0^l\left[\sin(n+1)\frac{\pi x}{l}-\sin(n-1)\frac{\pi x}{l}\right]dx$$

$$=\frac{1}{l}\left[-\frac{-\cos(n+1)\frac{\pi x}{l}}{(n+1)\frac{\pi}{l}}+\frac{\cos(n-1)\frac{\pi x}{l}}{(n-1)\frac{\pi}{l}}\right]_0^l$$

$$=\frac{1}{\pi}\left[\left\{-\frac{\cos(n+1)}{n+1}+\frac{\cos(n-1)\pi}{n-1}\right\}-\left\{-\frac{1}{n+1}+\frac{1}{n-1}\right\}\right]$$

$$=\frac{1}{\pi}\left[-\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]$$

When n is odd,

$$a_n=\frac{1}{\pi}\left[\frac{-1}{n+1}+\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]=0$$

When n is even,

$$a_n=\frac{1}{\pi}\left[\frac{1}{n+1}-\frac{1}{n-1}+\frac{1}{n+1}-\frac{1}{n-1}\right]=\frac{-4}{\pi(n+1)(n-1)}$$

$$\therefore \qquad \sin\left(\frac{\pi x}{l}\right)=\frac{2}{\pi}-\frac{4}{\pi}\left[\frac{\cos(2\pi x/l)}{1.3}+\frac{\cos(4\pi x/l)}{3.5}+\ldots\right]$$

Ex. 9: Obtain the half-range sine series for e^x in $0 < x < 1$

[JNTU 2000S, 2003S]

Sol. The half-range sine series of e^x is given by

$$e^x=\sum_{n=1}^{\infty}\sin n\pi x, \quad \text{since } l=1$$

Here $\qquad b_n = 2\int_0^1 e^x \sin n\pi x\, dx$

$$=2\left[\frac{e^x}{n^2\pi^2+1}(\sin n\pi x-n\pi\cos n\pi x)\right]_0^1$$

$$=2\left[\frac{e}{n^2\pi^2+1}(0-n\pi\cos n\pi)-\frac{1}{n^2\pi^2+1}(0-n\pi)\right]$$

$$=2\left[\frac{1}{n^2\pi^2+1}(n\pi-n\pi e\cos n\pi)\right] \quad =\frac{2n\pi}{n^2\pi^2+1}\left[1-e(-1)^n\right]$$

Hence $e^x=\pi\left[\frac{2(1+e)}{\pi^2+1}\sin\pi x+\frac{2(1-e)}{4\pi^2+1}\sin 2\pi x+\frac{3(1+e)}{9\pi^2+1}\sin 3\pi x+\ldots\right]$

Ex. 10: Expand $\cos\pi x$ in (0, 1) as Fourier sine series

OR

Show that in the interval (0, 1), $\cos\pi x=\frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2n\pi x$

[JNTU 2004S(Set No. 3)]

Sol. Let $f(x)=\cos\pi x=\sum_{n=1}^{\infty}b_n\sin n\pi x \;\; (\because l=1)$ ...(1)

Then $\quad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx, \quad$ where $l=1$

$$\therefore \qquad b_n = 2\int_0^1\cos\pi x\sin n\pi x\; dx=\int_0^1 2\sin n\pi x\,\cos\pi x\; dx \qquad ...(2)$$

$$= \int_0^1[\sin(n+1)\pi x+\sin(n-1)\pi x]dx$$

$$= \left[-\frac{\cos(n+1)\pi x}{(n+1)\pi}-\frac{\cos(n-1)\pi x}{(n-1)\pi}\right]_0^1 (n\neq 1)$$

$$= -\frac{1}{\pi}\left\{\left(\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}\right)-\left[\frac{1}{n+1}+\frac{1}{n-1}\right]\right\}$$

$$= -\frac{1}{\pi}\left\{\left[\frac{(-1)^{n+1}}{n+1}+\frac{(-1)^{n-1}}{n-1}\right]-\left[\frac{1}{n+1}+\frac{1}{n-1}\right]\right\}(n \neq 1)$$

$$= -\frac{1}{\pi}\left\{(-1)^{n+1}\left[\frac{1}{n+1}+\frac{1}{n-1}\right]-\left[\frac{1}{n+1}+\frac{1}{n-1}\right]\right\}$$

$$= -\frac{1}{\pi}\left(\frac{1}{n+1}+\frac{1}{n-1}\right)\left[(-1)^{n+1}-1\right]$$

$$= -\frac{1}{\pi}\cdot\frac{2n}{n^2-1}\left[(-1)^{n+1}-1\right] \quad (n \neq 1)$$

If n is even

$$b_n = \frac{-1}{\pi}\cdot\frac{2n}{n^2-1}(-2) = \frac{4n}{\pi(n^2-1)}$$

If n is odd

$$b_n = \frac{-1}{\pi}\cdot\frac{2n}{n^2-1}(1-1) = 0$$

If $\quad n = 1$, then

$$b_1 = \int_0^1 (2\sin \pi x \cos \pi x)dx \qquad \text{[Putting } n = 1 \text{ in (2)]}$$

$$= \int_0^1 \sin 2\pi x \, dx = -\frac{1}{2\pi}(\cos 2\pi x)_0^1$$

$$= -\frac{1}{2\pi}(\cos 2\pi - \cos 0) = 0$$

Substituting the values of b's in (1), we get

$$\cos \pi x = \sum_{n=2,4,6,\dots}^{\infty} \frac{4n}{\pi(n^2-1)} \sin n\pi x$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2n}{4n^2-1}\sin 2n\pi x \qquad \text{(Replacing } n \text{ with } 2n\text{)}$$

$$= \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{n}{4n^2-1}\sin 2n\pi x$$

which is the required Fourier series.

EXERCISE 10 (D)

1. Expand $f(x) = x^2 - 2$ as a Fourier series in the interval $(-2, 2)$.

[JNTU Dec. 2002 (Set No. 4)]

OR

Find the Fourier series to represent $f(x) = x^2 - 2$, when $-2 \leq x \leq 2$

[JNTU 2003S (Set No. 4), 2005S, 2007 (Set No. 2)]

2. Find the Fourier series expansion for the function $f(x) = x - x^2$ in $(-1, 1)$.

[JNTU 2003, 2004, 2005 (Set No. 1), 2006 (Set No. 4)]

3. Find the Fourier series of the following function :

$$f(x) = 0, -2 < x < 0$$
$$= 1, 0 < x < 2$$

4. Obtain the Fourier series for the function $f(x)$ given by

(*i*) $f(x) = \begin{cases} 0 \text{ when } 0 < x < l \\ a \text{ when } l < x < 2l \end{cases}$ (*ii*) $f(x) = \begin{cases} 0 \text{ if } -5 < x < 0 \\ 3 \text{ if } 0 < x < 5 \end{cases}$

5. Obtain the Fourier series for the function

$$f(x) = \pi x, \text{ when } 0 \le x \le 1$$
$$= \pi(2 - x), \text{ when } 1 \le x \le 2.$$

[JNTU 2007 (Set No.3)]

6. Obtain the Fourier series for $f(x) = x^2$ in (0, 3).

7. (*a*) Find the Fourier series of e^{-x} in $[-l, l]$.
(*b*) Expand $f(x) = e^{-x}$ as a Fourier series in $(-1, 1)$. [JNTU 2003S (Set No. 2)]

8. Find Fourier series to represent e^{ax} in $(-l, l)$.

9. If $f(x) = \frac{x}{l}$, for $0 < x < l$

$$= \frac{2l - x}{l}, \text{ for } l < x < 2l.$$

Show that $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left(\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \frac{1}{5^2}\cos\frac{5\pi x}{l} + \ldots\right)$.

10. Find the Fourier series of the function

(*i*) $f(x) = \begin{cases} 0 \text{ if } -2 < x < -1 \\ k \text{ if } -1 < x < 1 \\ 0 \text{ if } 1 < x < 2 \end{cases}$ (*ii*) $f(x) = \begin{cases} 0 \text{ if } -2 \le x \le -1 \\ 1 + x \text{ if } -1 \le x \le 0 \\ 1 - x \text{ if } 0 \le x \le 1 \\ 0 \text{ if } 1 \le x \le 2 \end{cases}$

11. Find the half-range cosine series for the function $f(x) = x$ in $(0, l)$.

12. Obtain the half-range cosine and sine series for x in (0, 2). [JNTU 2003]

13. Find the half-range cosine series for the function $f(x) = (x-1)^2$ in the interval $0 < x < 1$. Hence deduce that

(*i*) $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$

(*ii*) $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$ [JNTU 2001, 2005S, 2007 (Set No. 1)]

14. Obtain the half-range sine series for $x - x^2$ in (0, 1).

(OR)

Find the half-range sine series for the funtion $f(t) = t - t^2,\ 0 < t < 1$

[JNTU 2004S (Set No. 1)]

15. Express $f(x) = lx - x^2$ as a half range sine series in $(0, l)$. Hence show that

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} \ldots..$$

16. Find the Fourier sine series to represent the function $f(x) = \sin x,\ 0 < x < \pi$

17. Express $f(x) = x^3$ as half-range cosine series in (0, L) **[JNTU 2004S (Set No. 1)]**

18. Represent $f(x) = x^2$ in $0 < x < L$ by Fourier sine series **[JNTU 2004S (Set No. 2)]**

19. Represent $f(x) = 2x - 1$ in $0 < x < 1$ by Fourier cosine series.

20. Obtain the half-range sine series for the function

$$f(x) = \frac{1}{4} - x, \text{ when } 0 < x < \frac{1}{2}$$
$$= x - \frac{3}{4} \text{ when } \frac{1}{2} < x < 1.$$

[JNTU 2003S (Set No. 1)]

21. Find the half-range cosine series for the function $f(x)$ given by

$$f(x) = kx, \text{ when } 0 \le x \le \frac{l}{2}$$
$$= k(l - x), \text{ when } \frac{l}{2} \le x \le l$$

[JNTU 2005S (Set No. 3)]

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

ANSWERS

1. $x^2 - 2 = \frac{-2}{3} - \frac{16}{\pi^2}\left(\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\pi x + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \frac{1}{4^2}\cos 2\pi x + \ldots\right)$

2. $x - x^2 = \frac{-1}{3} + \frac{4}{\pi^2}\left(\cos\frac{\pi x}{1^2} - \frac{\cos 2\pi x}{2^2} + \cos\frac{3\pi x}{3^2} - \ldots\right)$
$+ \frac{2}{\pi}\left(\sin\pi x - \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x - \ldots\right)$

3. $f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\sin\frac{\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} + \frac{1}{5}\sin\frac{5\pi x}{2} + \ldots\right)$

4. (i) $f(x) = \frac{a}{2} - \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin(2n-1)\frac{\pi x}{l}$ (ii) $f(x) = \frac{3}{2} + \frac{6}{\pi}\sum_{n=1,3,5,..}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{5}\right)$

5. $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right)$

6. $x^2 = \left(\frac{18}{\pi} - \frac{72}{\pi^3}\right)\sin\frac{\pi x}{3} - \frac{18}{2\pi}\sin\frac{2\pi x}{3} + \left(\frac{18}{3\pi} - \frac{72}{27\pi^3}\right)\sin\frac{3\pi x}{3} + \ldots$

7. (a) $e^{-x} = \sinh l\left[\frac{1}{l} + \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2 + l^2}\left(2l\cos\frac{n\pi x}{l} + 2n\pi\sin\frac{n\pi x}{l}\right)\right]$

8. $e^{ax} = \sinh al\left[\frac{1}{al} + 2al\sum_{n=1}^{\infty}\frac{(-1)^n}{a^2l^2 + n^2\pi^2}\cos\frac{n\pi x}{l} + 2\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{a^2l^2 + n^2\pi^2}\sin\frac{n\pi x}{l}\right]$

10. (i) $f(x) = \frac{k}{2} + \frac{2k}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\cdot\cos\left[\frac{n\pi x}{2}\right]$ (ii) $f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1 - \cos\frac{n\pi}{2}\right)\cdot\cos\left(\frac{n\pi x}{2}\right)$

11. $x=\frac{l}{2}-\frac{4l}{\pi^2}\left[\cos\frac{\pi 4}{l}+\frac{1}{3^2}\cos\frac{3\pi x}{l}+\frac{1}{5^2}\cos\frac{5\pi x}{l}+\ldots\right]$.

12. $x=1-\frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+\ldots\right]$;

$x=\frac{4}{\pi}\left[\sin\frac{\pi x}{2}-\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}-\frac{1}{4}\sin\frac{4\pi x}{2}+\ldots\right]$.

13. $(x-1)^2=\frac{1}{3}+\frac{4}{\pi^2}\left(\frac{1}{1^2}\cos\pi x+\frac{1}{2^2}\cos 2\pi x+\frac{1}{3^2}\cos 3\pi x+\frac{1}{4^2}\cos 4\pi x+\ldots\right)$.

14. $x-x^2=\frac{8}{\pi^3}\left(\frac{1}{1^3}\sin\pi x+\frac{1}{3^3}\sin 3\pi x+\frac{1}{5^3}\sin 5\pi x+\ldots\right)$.

15. $lx-x^2=\frac{8l^2}{\pi^3}\left(\sin\frac{\pi x}{l}+\frac{1}{3^3}\sin\frac{3\pi x}{l}+\frac{1}{5^3}\sin\frac{5\pi x}{l}+\ldots\right)$

16. $f(x)=\sin x$

17. $x^3=\frac{L^3}{4}+\sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{-3L^3}{2n^2\pi^2}+\frac{12L^3}{n^4\pi^4}\right)\cos\frac{n\pi x}{L}+\sum_{n=2,4,6,\ldots}^{\infty}\frac{3L^3}{2n^2\pi^2}\cos\frac{n\pi x}{L}$

18. $x^2=\frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left[\frac{2}{n^3\pi^2}\{(-1)^n-1\}-\frac{(-1)^n}{n}\right]\sin\frac{n\pi x}{L}$

19. $2x-1=-\frac{8}{\pi}\left[\frac{\cos\pi x}{1^2}+\frac{\cos 3\pi x}{3^2}+\frac{\cos 5\pi x}{5^2}+\ldots\ldots\right]$

20. $f(x)=\left(\frac{1}{\pi}-\frac{4}{\pi^2}\right)\sin\pi x+\left(\frac{1}{3\pi}+\frac{4}{3^2\pi^2}\right)\sin 3\pi x+\left(\frac{1}{5\pi}-\frac{4}{5^2\pi^2}\right)\sin 5\pi x+\ldots$

21. $f(x)=\frac{kl}{4}-\frac{8kl}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi x}{l}+\frac{1}{6^2}\cos\frac{6\pi x}{l}+\frac{1}{10^2}\cos\frac{10\pi x}{l}+\ldots\right)$

QUIZ

1. The trigonometrical series of $f(x)$ in the interval $(-\pi, \pi)$ is __________
2. If $x = a$ is a point of discontinuity then the Fourier series of $f(x)$ at $x = a$ is given by $f(x) =$ __________
3. In the Fourier series expansion of a function, the Fourier coefficient a_0 represents the __________ value of the function.
4. Conditions for expansion of a function in Fourier series are known as __________ conditions.
5. The rate of convergence of a Fourier series increases while the series is __________
6. A function $f(x)$ defined for $0 < x < \pi$ can be extended to an odd periodic function in $(-\pi, \pi)$ such that $f(-x) =$ __________
7. A function $f(x)$ defined for $0 < x < 2$ can be extended to an even periodic function in the interval $(-2, 2)$ such that $f(-x) =$ __________

8. Fourier series expansion of an even function in $(-c, c)$ has only __________ terms.
9. Fourier series expansion of an odd function in $(-l, l)$ has only __________ terms.
10. If $f(x)$ is an even function in $(-\pi, \pi)$, then the graph of $f(x)$ is symmetrical about the __________
11. If $f(x)$ is an odd function $(-l, l)$, then the graph of $f(x)$ is symmetrical about the __________
12. If $f(x)$ is an even function in the interval $(-l, l)$ then the value of $b_n =$ __________
13. If $f(x)$ is an odd function in $(-l, l)$ then the values of a_0 and a_n are __________
14. If $f(x) = x$ in $(-\pi, \pi)$ then the Fourier coefficient $a_2 =$ __________
15. If $f(x) = x^2$ in $(-l, l)$ then $b_1 =$ __________
16. If $f(x) = x^3$ in $(-l, l)$ then $a_n =$ __________
17. If $f(x) = \cos x$ in $(-\pi, \pi)$ then Fourier coefficient $b_n =$ __________
18. In the Fourier series expansion of $f(x) = |x|$ in $(-\pi, \pi)$ the value of $b_1 =$ __________
19. If $f(x) = |x|$ in $(-\pi, \pi)$ then $a_1 =$ __________
20. If $f(x) = |\sin x|$ in $(-l, l)$ then the Fourier coefficient $b_n =$ __________
21. If $f(x) = |\cos x|$ in $(-\pi, \pi)$ then $b_2 =$ __________
22. In the Fourier series expansion of $f(x) = x \sin x$ in $(-\pi, \pi)$ the __________ terms are absent.
23. If $f(x) = x \cos x$ in $(-\pi, \pi)$ then $b_1 =$ __________
24. If $f(x) = |\cos x|$ in $(-\pi, \pi)$ then $a_0 =$ __________
25. If $f(x) = x^2$ in $(-l, l)$ then $a_0 =$ __________
26. If $f(x)$ is a periodic function with period 2T then $b_n =$ __________
27. If $f(x) = \begin{cases} -1 \text{ when } -1 < x < 0 \\ 1 \text{ when } \ 0 < x < 1 \end{cases}$

    then $f(x)$ is an __________ function in $(-1, 1)$.
28. If $f(t) = \begin{cases} -t \text{ for } -\pi < t < 0 \\ t \text{ for } \ 0 < t < \pi \end{cases}$

    then $f(x)$ is an __________ function in $(-\pi, \pi)$.
29. If $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} \text{ in } -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi} \text{ in } \ 0 \le x \le \pi \end{cases}$

    then $f(x)$ is an __________ function.
30. If $f(x) = \begin{cases} 1 - x \text{ if } -\pi < x < 0 \\ 1 + x \text{ if } \ 0 < x < \pi \end{cases}$

    then $f(x)$ is an __________ function.

31. Fourier series for $f(x) = x$ in $(-\pi, \pi)$ is __________

32. Fourier series for $f(x) = x^2$ in $(-\pi, \pi)$ is __________

33. Fourier series for $f(x) = |x|$ in $(-\pi, \pi)$ is __________

34. Fourier series for $f(x) = 1-x^2$ in $(-1, 1)$ is __________

35. Fourier series for $f(x) = ax + b$ in $0 < x < 1$ is __________

36. The formulae for finding the half - range cosine series for the function $f(x)$ in $(0,l)$ are given by $a_0 =$ __________ and $a_n =$ __________

37. The half range sine series for $f(x) = 1$ in $(0, \pi)$ is __________

38. Fourier sine series for $f(x) = x$ in $(0, \pi)$ is __________

39. The half range sine series for $f(x) = e^x$ in $(0, \pi)$ is __________

40. In the half range cosine series of $f(x) = x \sin x$, $0 < x < \pi$, the value of $a_0 =$ __________

41. In the half range cosine series of $f(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 < x < 2 \end{cases}$, the value of $a_0 =$ __________

42. If $f(x) = x$ in $(0, 2\pi)$ then the Fourier coefficient $a_0 =$ __________

43. If $f(x) = \begin{cases} \cos x & \text{for } 0 < x < \dfrac{\pi}{2} \\ 0 & \text{for } \dfrac{\pi}{2} < x < \pi \end{cases}$ then $a_0 =$ __________

44. If $f(x) = \begin{cases} 0 & \text{in } -\pi < x \le 0 \\ x & \text{in } 0 < x < \pi \end{cases}$ then $a_0 =$ __________

45. The Fourier series expansion of $f(x) = \begin{cases} -x & \text{in } -4 \le x \le 0 \\ x & \text{in } 0 \le x \le 4 \end{cases}$ contains no __________ terms.

46. The Fourier series expansion of $f(x) = \begin{cases} -x^2 & \text{in } -\pi < x \le 0 \\ x^2 & \text{in } 0 \le x \le \pi \end{cases}$ contains no __________ terms.

47. Fourier series of the function $f(x) = \sin^3 x$ in $(-\pi,\pi)$ is __________

48. Fourier series of the function $f(x) = \cos^3 x$ in $(-\pi,\pi)$ is __________

49. Fourier series of $f(x) = \begin{cases} 0 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$ is __________

50. Fourier series of $f(x) = \begin{cases} -k & \text{for } -\pi < x < 0 \\ k & \text{for } 0 < x < \pi \end{cases}$ is __________

51. Fourier series of $f(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 1 & \text{for } 0 < x < \pi \end{cases}$ is __________

52. Fourier series of $f(x) = \begin{cases} a & \text{when } 0 < x < \pi \\ -a & \text{when } \pi < x < 2\pi \end{cases}$ is __________

ANSWERS

1. $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx$ 2. $f(x)=\frac{1}{2}[f(a-0)+f(a+0)]$ 3. mean

4. Dirichlet's 5. Integrating 6. $-f(x)$ 7. $f(x)$ 8. cosine

9. sine 10. y-axis 11. origin 12. 0 13. Zeros

14. 0 15. 0 16. 0 17. 0 18. 0

19. $-\frac{2}{\pi}$ 20. 0 21. 0 22. sine 23. 0

24. $\frac{4}{\pi}$ 25. $\frac{2l^2}{3}$ 26. $\frac{1}{T}\int_{\alpha}^{\alpha+2T} f(x) \sin\frac{n\pi x}{T}dx$ 27. odd

28. even 29. even 30. even 31. $2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-...\right)$

32. $\frac{\pi^2}{3}-4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\frac{\cos 3x}{3^2}-...\right)$ 33. $\frac{\pi}{2}-\frac{4}{\pi}\left(\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+...\right)$

34. $\frac{2}{3}+\frac{4}{\pi^2}\left(\cos \pi x-\frac{1}{2^2}\cos 2\pi x+\frac{1}{3^2}\cos 3\pi x-...\right)$ 35. $2a\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}\sin n\pi x$

36. $\frac{2}{l}\int_0^l f(x)dx,\ \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx$ 37. $\frac{4}{\pi}\left(\sin x+\frac{\sin 3x}{3}+\frac{\sin 5x}{5}+...\right)$

38. $2\left(\sin x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-...\right)$ 39. $\frac{2}{9}\sum_{n=1}^{\infty}\frac{n}{n^2+1}\left[1+(-1)^{n+1}e^{\pi}\right]\sin x$

40. 2 41. 1 42. 2π 43. $\frac{2}{\pi}$ 44. $\frac{\pi}{2}$

45. sine 46. cosine 47. $\frac{3}{4}\sin x-\frac{1}{4}\sin 3x$

48. $\frac{3}{4}\cos x+\frac{1}{4}\cos 3x$ 49. $\frac{1}{2}+\frac{2}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+...\right)$

50. $\frac{4k}{\pi}\sum_{n=1,3,5,..}^{\infty}\frac{\sin nx}{n}$ 51. $\frac{4}{\pi}\sum_{n=1,3,5,..}^{\infty}\frac{\sin nx}{n}$ 52. $\frac{4a}{\pi}\sum_{n=1,3,5,..}^{\infty}\frac{\sin nx}{n}$