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Mathematics

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Mathematics

2018/2019
For Geophysics students

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College of Remote Sensing and Geophysics
Department of Geophysics
Title of the course Mathematics
Level : 2nd Stage, 1st Semester
Hours: 2 Hours / Week

References:

1- Mathematical Methods (third Edition 2008) by Dr. T.K.v. Iyengar, Dr. B.krishna Gandhi

Matrices

1

Definition matrix: Asystem of mn number (real or complex) arranged in the form of an ordered set of m rows. each row consisting of an ordered set of n numbers betwen [ ] or ( ) or ‖ ‖ is called a matrix of order of type $m \times n$.
each of mn numbers constituting the $m \times n$ matrix is called an element of the matrix.
Thus we write a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} = [a_{ij}]_{m \times n} \quad \text{where } 1 \le i \le m, \; 1 \le j \le n$$

Som Types of matrices

1- Square matrix

if $A = [a_{ij}]_{m \times n}$ and $m = n$ then A is called a Square matrix. Asquare A order $n \times n$ is Sometimes called as a n-rowed matrix A or Simply a square matrix of order n

Ex $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is 2^{nd} order matrix.

Ex
$$A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 4 \\ -1 & 0 & 3 \end{bmatrix}$$ is a square matrix of order 3

2- rectangular matrix

A matrix which is not a square matrix is called a rectangular matrix.

Ex

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$$ is 2×3 matrix

3 - row matrix

A matrix of order $1\times m$ is called row matrix

Ex

$A = [1 \;\; 2 \;\; 3]$, Ex $A = [3 \;\; -1 \;\; 2 \;\; 7]_{1\times4}$

4- Column matrix

A matrix of order $n\times1$ is called a column matrix

Ex

$$A = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{3\times1}$$

Ex

$$A = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}_{4\times1}$$

row and column matrices are called row and Column vectors.

5 - Identity matrix

if $A = [a_{ij}]_{n\times n}$ such that $a_{ii} = 1$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$. then A is called identity matrix. it is denoted by I_n.

Ex

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$$

Ex

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times4}$$

Ex

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$$

6- Zero matrix

3

if $A=[a_{ij}]_{m\times n}$ such that $a_{ij}=0\ \forall i,j$, then A is called a Zero matrix of a null matrix. It is denoted by O or $O_{m\times n}$

Ex

$$A=O=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$$

Ex

$$A=O=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4}$$

7. Diagonal elements of square matrix (principle diagonal)

Def. In a matrix $A=[a_{ij}]_{m\times n}$ the elements a_{ij} of A for which $i=j$ ($a_{11}, a_{22}, \dots, a_{nn}$) are called the diagonal elements of A.

The line along which the diagonal elements lie is called the principle diagonal of A

8- Diagonal matrix

A square matrix all of whose elements except those in leading diagonal are Zero is called diagonal matrix. If $d_1, d_2, \dots, d_n$ are diagonal elements of a diagonal matrix A. then A is written as $A=\text{diag}(d_1, d_2, \dots, d_n)$

Ex

$$A=\text{diag}(3,1,-2)=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

9- Scalar matrix

A diagonal matrix whose leading diagonal elements are equal is called a scalar matrix.

Ex

$$A=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4

10- Equal matrices

Def. Two matrices $A=[a_{ij}]$, $B=[b_{ij}]$ are said to be equal if and only if

i) A and B are of the same type (or order)

ii) $a_{ij}=b_{ij}$ for every i and j

Ex

$$A=\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}, \quad A=B$$

11- The Transpose of a matrix

def. the matrix obtained from any given matrix A, by inter changing its rows and columns is called The transpose of A, it is denoted by A' or A^T

if $A=[a_{ij}]_{m\times n}$ then the transpose of A is $A'=[b_{ji}]_{n\times m}$ wher $b_{ji}=a_{ij}$ also $(A')'=A$

Ex

$$A=\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 4 \\ -1 & 0 & 3 \end{bmatrix}_{3\times 3}, \quad A^T=A'=\begin{bmatrix} 2 & 0 & -1 \\ 3 & -2 & 0 \\ -1 & 4 & 3 \end{bmatrix}$$

Not

if A' and B' be transposs of A and B respectively then

i) $(A')'=A$

ii) $(A+B)'=A'+B'$, A and B being of the same order

iii) $(KA)'=KA'$, k is scalar

iv) $(AB)'=B'A'$, A and B being conformable for multiplication.

5

12 – Triangular matrix

A square matrix all of whose elements below the leading diagonal are zero is called an upper triangular matrix. A square matrix all of whose elements above the leadings zero is called a lower triangular matrix

Ex

$$A = \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & 4 & 2 & 1 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$ is an upper triangular matrix

Ex

$$A = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 5 & 3 & 0 & 0 & 0 \\ -4 & 6 & 0 & 0 & 0 \\ -2 & 1 & -8 & 5 & 0 \\ 2 & 0 & 4 & 1 & 6 \end{bmatrix}$$ is a lower triangular matrix

Ex

$$A = \begin{bmatrix} 1 & 4 & -6 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$ upper triangular matrix

E

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 7 & 2 & 0 \\ 1 & 4 & -1 \end{bmatrix}$$ Lower triangular matrix

13 – Symmetric matrix

A square matrix $A = [a_{ij}]$ is said to be symmetric if $a_{ij} = a_{ji}$ for every i and j

Ex

$$A = \begin{bmatrix} 2 & 3 & -5 \\ 3 & 1 & 2 \\ -5 & 2 & -1 \end{bmatrix}$$

14- Skew-symmetric matrix 6

A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if $a_{ij} = -a_{ji}$ every i and j

Ex

$$A = \begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & 2 \\ 5 & -2 & 0 \end{bmatrix}$$

Note 1- A is a skew-symmetric matrix $\Longleftrightarrow$ $A = -A'$ or $A' = A$

2- every diagonal element of a skew-symmetric matrix is necessarly zero.

$a_{ij} = -a_{ji} \Longrightarrow a_{ii} = 0$

Ex

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$ is a symmetric matrix

and $$A = \begin{bmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{bmatrix}$$ is a skew-symmetric matrix

15 Multiplication of matrix by a scalar

let A be matrix, the matrix obtained by Multiplying every of A by k, a scalar is called the product of A by k and is denoted by kA or Ak

if $A=[a_{ij}]_{m\times n}$ then $kA=[ka_{ij}]_{m\times n}=k[a_{ij}]_{m\times n}$
$=kA$

Properties

i $0A=0$ (null matrix) $(-1)A=-A$ called the negative of A

ii $k_1(k_2A)=(k_1k_2)A=k_2(k_1A)$ where k_1, k_2 are scalars

iii A is symmetric $\Rightarrow kA$ is symmetric

iv $(kA)'=kA'$

v A is skew-symmetric $\Rightarrow kA$ is skew-symmetric

vi $kA=0 \Rightarrow A=0$ if $k\neq 0$

vii $k_1A=k_2A$ and A is not null matrix $\Rightarrow k_1=k_2$

16- Sum of matrices or matrix addition

let $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$ be two matrices. the matrix $C=[c_{ij}]_{m\times n}$ where

$[c_{ij}]=[a_{ij}]+[b_{ij}]$ is called the sum of the

matrices A and B. the sum of A and B is denoted by $A+B$

$[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}=[a_{ij}+b_{ij}]_{m\times n}$

and $[a_{ij}+b_{ij}]_{m\times n}=[a_{ij}]_{m\times n}+[b_{ij}]_{m\times n}$

17 The difference of two matrices 8

if A and B two matrices of Same type (order) then $A + (-B)$ is taken as $A - B$

Ex let $[A] = \begin{bmatrix} 2 & 0 & -1 \\ 7 & 1 & 4 \end{bmatrix}_{2\times3}$, $[B] = \begin{bmatrix} -3 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}_{2\times3}$

Then

$$[C] = [A] + [B] = \begin{bmatrix} 2 & 0 & -1 \\ 7 & 1 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$[C] = \begin{bmatrix} -1 & 0 & -2 \\ 7 & 2 & 6 \end{bmatrix}_{2\times3}$$

and

$$[A] - [B] = \begin{bmatrix} 2 & 0 & -1 \\ 7 & 1 & 4 \end{bmatrix} - \begin{bmatrix} -3 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 7 & 0 & 2 \end{bmatrix}_{2\times3}$$

multiplication by a constant

Ex

$$[A] = \begin{pmatrix} 2 & -1 & 4 \\ 0 & 5 & -2 \end{pmatrix} \quad \text{find } 2A$$

$$2A = 2\begin{pmatrix} 2 & -1 & 4 \\ 0 & 5 & -2 \end{pmatrix} = \begin{pmatrix} 4 & -2 & 8 \\ 0 & 10 & -4 \end{pmatrix}$$

18- matrix multiplication 9

let $A = [a_{ik}]_{m\times n}$, $B = [b_{kj}]_{n\times p}$ then the matrix $C = [c_{ij}]_{m\times p}$

where

$$c_{ij} = \sum_{k=1}^{n} a_{ij}$$

called the product of the matrices A and B in that order and we write

$C = AB$

In the product AB ; the matrix A is called the pre-factor and B the post-factor

if the number of columns of A is equal to the number of rows in B then then the matrices said to be conformable for multiplication in that order

Ex

$$A = \begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix}_{3\times 3} \qquad B = \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}_{3\times 1}$$ find A.B

$$A\cdot B = \begin{pmatrix} (3)(2)+(4)(5)+(0)(-1) \\ (-2)(2)+(6)(5)+(-3)(-1) \\ (7)(2)+(-4)(5)+(1)(-1) \end{pmatrix} = \begin{pmatrix} 26 \\ 29 \\ -7 \end{pmatrix}_{3\times 1}$$

Ex

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} , B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

$$A\cdot B = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2\times 2 + 3\times 0 & (2)(3) + 3\times 1 \\ 1\times 2 + 0\times 0 & 1\times 3 + 0\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 9 \\ 1 & 0 \end{bmatrix}$$

But $B \cdot A = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$ 10

$$= \begin{bmatrix} 2\times 2 + 3\times 1 & 2\times 3 + 3\times 0 \\ 0\times 2 + 1\times 1 & 0\times 3 + 1\times 0 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 1 & 0 \end{bmatrix}$$

$$B \cdot A \neq AB$$

19- Positive integral powers of square matrices

let A b a square matrix. then A^2 is defined as $A \cdot A$

$A^2 A = (A \cdot A) A = A A^2$

$A^2 A = A A^2 = A^3$

Similarly we have $A A^{m-1} = A^{m-1} A = A^m$

where m is a positive integer.

$A^m \cdot A^n = A^{m+n} = (A^m)^n = A^{mn}$

Note $I^n = I$, $O^n = O$

20- Trace of A square matrix

let $A = [a_{ij}]_{m \times n}$ the trace of the square matrix A is defined as

$\sum_{i=1}^{n} a_{ii}$ and is denoted by $[trA]$

$$trA = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

properties

if A and B are square matrices of order n and λ is any scalar. then

i- $tr(\lambda A) = \lambda \, trA$

ii- $tr(A+B) = trA + trB$

iii- $tr(A \cdot B) = tr(BA)$

Ex $A=\begin{bmatrix} -3 & 0 \\ 7 & -4 \end{bmatrix}$ $B=\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix}$, $C=\begin{bmatrix} 1 & 0 \\ -2 & -4 \end{bmatrix}$

find $2A-3B+4C$

$$2A-3B+4C = 2\begin{bmatrix} -3 & 0 \\ 7 & -4 \end{bmatrix} - 3\begin{bmatrix} 2 & -1 \\ -7 & 4 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ -2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} -6 & 0 \\ 14 & -8 \end{bmatrix} - \begin{bmatrix} 6 & -3 \\ -21 & 12 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ -8 & -16 \end{bmatrix}$$

$$= \begin{bmatrix} -6+6+4 & 0+3+0 \\ 14+21-8 & -8-12-16 \end{bmatrix} = \begin{bmatrix} -8 & 3 \\ 27 & -36 \end{bmatrix}$$

21-

orthogonal matrix.

A square matrix A is said to be orthogonal matrix if $A \cdot A' = A' \cdot A = I$

Ex

is the matrix $A=\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}$ orthogonal

Solution

$$A=\begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$$

$$A \cdot \bar{A} = \begin{bmatrix} 2 & -3 & 1 \\ 4 & 3 & 1 \\ -3 & 1 & 9 \end{bmatrix} \begin{bmatrix} 2 & 4 & -3 \\ -3 & 3 & 1 \\ 1 & 1 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4+9+1 & 8-9+1 & -6-3+9 \\ 8-9+1 & 16+9+1 & -12+3+9 \\ -6-3+9 & -12+3+9 & 9+1+81 \end{bmatrix} = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 91 \end{bmatrix} \neq I$$

the matrix A is not orthogonal

Ex Show that A is orthogonal

$$A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Solution

$$A = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \quad A^T = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$$A \cdot A^T = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$$

Ex prove the matrix is orthogonal

$$A = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$$

Solution

$$A^T = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{9}+\frac{1}{9}+\frac{4}{9} & -\frac{4}{9}+\frac{2}{9}+\frac{2}{9} & -\frac{2}{9}-\frac{2}{9}+\frac{4}{9} \\ -\frac{4}{9}+\frac{2}{9}+\frac{2}{9} & \frac{4}{9}+\frac{4}{9}+\frac{1}{9} & \frac{2}{9}-\frac{4}{9}+\frac{2}{9} \\ -\frac{2}{9}-\frac{2}{9}+\frac{4}{9} & \frac{2}{9}-\frac{4}{9}+\frac{2}{9} & \frac{1}{9}+\frac{4}{9}+\frac{4}{9} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Determinant of a Square matrix

if $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

then $|A|$ or $\det(A)$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}$$

Ex

- if $A_{1\times 1}$ $\quad A = [2] \Rightarrow \det(A) = |2| = 2$

  $A = [-5] \Rightarrow \det(A) = |-5| = -5$

- $A_{2\times 2}$ $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

  $\det(A) = ad - bc$

  $A = \begin{bmatrix} 3 & 1 \\ 4 & -2 \end{bmatrix}$ then $|A| = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = 3 \times -2 - 1 \times 4 = -10$

- $A_{3\times 3}$ $\quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$\begin{matrix} \times & a_{11} & a_{12} & a_{13} & - \\ \times & a_{21} & a_{22} & a_{23} & - \\ \times & a_{31} & a_{32} & a_{33} & - \end{matrix}$$

$|A| = a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31}$

$+ a_{21} \cdot a_{32} \cdot a_{13} - a_{13} \cdot a_{22} \cdot a_{31}$

$- a_{23} \cdot a_{32} \cdot a_{11} - a_{33} \cdot a_{21} \cdot a_{12}$

or

$$\begin{array}{ccccc} + & + & + & - & - & - \end{array}$$

$$A = \left| \begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array} \right|$$

$|A| = a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32}$
$- a_{13} \cdot a_{22} \cdot a_{31} - a_{11} \cdot a_{23} \cdot a_{32} - a_{12} \cdot a_{21} \cdot a_{33}$

Ex Find $|A|$ $A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 3 & -1 & 3 \end{bmatrix}$

$$\begin{array}{ccccc} + & + & + & - & - & - \end{array}$$

$$|A| = \left| \begin{array}{ccc|cc} 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 3 & -1 & 3 & 3 & -1 \end{array} \right|$$

$|A| = 2*2*3 + 1*1*3 + 2*1*-1$
$- 2*2*3 - 2*1*-1 - 1*1*3 \quad = 0$

Definition: A square matrix A is said to be singular if $|A| = 0$
if $|A| \neq 0$, then A is said to be non-singular. thus only non-singular matrixes possess inverses.

Note if A, B are non-singular, then AB the product is also non-singular

Properties of Determinants 15

1- if two rows (or columns) are identical then $\det(A) = 0$

ex $\begin{vmatrix} 1 & 2 & 3 \\ -1 & 4 & 7 \\ -1 & 4 & 7 \end{vmatrix} = 0$ ex $\begin{vmatrix} 2 & -5 & 1 \\ 3 & 0 & 2 \\ 2 & -5 & 1 \end{vmatrix} = 0$

2 - interchanging any two rows (or columns) changes the sign of determinant

ex $\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 0 \\ 2 & 5 & 7 \end{vmatrix} = - \begin{vmatrix} 3 & 1 & 0 \\ 2 & -1 & 3 \\ 2 & 5 & 7 \end{vmatrix}$

3 - $|A^T| = |A|$

4- if arrow (or columns) is multiplied by k (seetor) the the determinant multiplied by k.

Ex $\begin{vmatrix} 2 & -4 & 18 \\ 1 & 0 & 6 \\ 5 & -1 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & -2 & 9 \\ 1 & 0 & 6 \\ 5 & -1 & 3 \end{vmatrix} = 2 * 3 \begin{vmatrix} 1 & -2 & 3 \\ 1 & 0 & 2 \\ 5 & -1 & 1 \end{vmatrix}$

5 $|AB| = |A||B|$

Minors and Cofactors of a square matrix

let $A=[a_{ij}]_{n\times n}$ be square matrix, when from A elements of ith row and jth Column are deleted the determinant of (n-1) rowed matrix M_{ij} is called the minor of a_{ij} of A and is denoted by $|M_{ij}|$ the signed minor
$(-1)^{i+j}|M_{ij}|$ is called the cofactor of a_{ij} and is denoted by A_{ij}

~~adjoint of a square matrix~~
~~The adjoint of matrix A is defined by~~

if $A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$|A| = a_{11}|M_{11}| - a_{12}|M_{12}| + a_{13}|M_{13}|$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Note Determinant of the square matrix A can be defined as

$$|A| = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$
$$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$$

17

or $|A| = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

$= a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$

$= a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

Adjoint of square matrix

let A be a square matrix of order n. the transpose of the matrix got from A by replacing the elements of A by the corresponding cofactors is called the adjoint of A and is denoted by

adj A.

if $|A| \neq$ then $A^{-1} = \frac{1}{|A|}(\text{adj} A)$

Ex Define adjoint of a matrix and hence find A^{-1} by using adjoint of A where

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

Solution

$$\det(A) = 1\begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} - 1\begin{vmatrix} 1 & -3 \\ 2 & -4 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix}$$

$= -8 \neq 0$

A is non-singular $\Rightarrow A^{-1}$ exists

Calculation of cofactors 18

row 1

Cofactor of $1 = (-1)^{1+1} \begin{vmatrix} 3 & -3 \\ -4 & -4 \end{vmatrix} = -12 - 12 = -24$

Cofactor of $1 = (-1)^{1+2} \begin{vmatrix} 1 & -3 \\ -2 & -4 \end{vmatrix} = -(-4-6) = 10$

Cofactor of $3 = (-1)^{1+3} \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = (-)(-4+6) = 2$

Row 2

Cofactor of $1 = (-1)^{2+1} \begin{vmatrix} 1 & 3 \\ -4 & -4 \end{vmatrix} = -(-4+12) = -8$

Cofactor of $3 =$
$= (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ -2 & -4 \end{vmatrix} = (-)(-4+6) = 2$

Cofactor of $-3 = (-1)^{2+3} \begin{vmatrix} 1 & 1 \\ -2 & -4 \end{vmatrix} = -1(-4+2) = 2$

Row 3

Cofactor of $-2 = (-1)^{3+1} \begin{vmatrix} 1 & 3 \\ 3 & -3 \end{vmatrix} = 1(-3-9) = -12$

Cofactor of $-4 = (-1)^{3+2} \begin{vmatrix} 1 & 3 \\ 1 & -3 \end{vmatrix} = -1(-3-3) = 6$

Cofactor of $-4 = (-1)^{3+3} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = 1(3-1) = 2$

the matrix formed the cofactors of 19
elements of A be

$$B = \begin{bmatrix} -24 & 10 & 2 \\ -8 & 2 & 2 \\ -12 & 6 & 2 \end{bmatrix}$$

$$\text{adj } A = B^T = \begin{bmatrix} -24 & -8 & -12 \\ 10 & 2 & 6 \\ 2 & 2 & 2 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Ex let $A=\begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$ 20

find adj A

Solution

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} = -18$$

$$A_{21} = (-1)^{2+1} \begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} = -6$$

$$A_{31} = (-1)^{3+1} \begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} ----$$

$$\operatorname{adj} A = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & -28 \end{bmatrix}$$

Ex Find the adjoint and inverse of

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

Solution

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

where A_{ij} are the cofactors of the elements a_{ij} thus minors of a_{ij} are

$$M_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 4 \end{vmatrix} = 10 \quad M_{12} = \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = 15$$

$$M_{13} = \begin{vmatrix} 4 & 3 \\ 1 & 2 \end{vmatrix} = 5 \ , \ M_{21} = \begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = 4$$

$$M_{22} = \begin{vmatrix} 2 & 4 \\ 1 & 4 \end{vmatrix} = 4 \ , \ M_{23} = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1$$

$$M_{31} = \begin{vmatrix} 3 & 4 \\ 3 & 1 \end{vmatrix} = -9 \ , \ M_{32} = \begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = -14$$

$$M_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = -6$$

$$\text{Cofactor } A_{ij} = (-1)^{i+j} M_{ij}$$

$$\text{adjoint of } A = \begin{bmatrix} 10 & -15 & 5 \\ -4 & 4 & -1 \\ -9 & 14 & -6 \end{bmatrix}^T$$

$$= \begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

$$|A| = 2(12-2) - 3(16-1) + 4(8-3)$$ 22

$$= 20 - 45 + 20 = 40 - 45 = -5 \neq 0$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A = \frac{1}{-5}\begin{bmatrix} 10 & -4 & -9 \\ -15 & 4 & 14 \\ 5 & -1 & -6 \end{bmatrix}$$

Ex

Find A^{-1} if $A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$

Solution

1- $|A| = -94 \neq 0$ check

2- $\operatorname{cof} A = \begin{bmatrix} -18 & 17 & -6 \\ -6 & -10 & -2 \\ -10 & -1 & 28 \end{bmatrix}$

3- $\operatorname{adj} A = [\operatorname{cof} A]^T = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$

4- $A^{-1} = \frac{\operatorname{adj} A}{|A|} = \frac{-1}{94}\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$

To check $A \cdot A^{-1} = A^{-1} A = I$

$$\begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix} * \frac{-1}{94}\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex Find $|A|$ of A 23

$$A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 3 \\ 1 & 0 & 2 & 3 \\ 0 & 4 & 0 & 0 \end{bmatrix}$$

Solution

$$|A| = 0(-1)^{4+1} \begin{vmatrix} 0 & 0 & 1 \\ 2 & 0 & 3 \\ 0 & 2 & 3 \end{vmatrix}$$

$$+ 4(-1)^{4+2} \begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

$$+ 0(-1)^{4+3} \begin{vmatrix} 2 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 0 & 3 \end{vmatrix}$$

$$+ 0(-1)^{4+4} \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

$$|A| = 0 + 4 \begin{vmatrix} 2 & 0 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

=

$$= \cancel{4\left(0(-1)^{1+3} \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} + 0(-1)^{2+2}\right)}$$

24

$$|A| = 4\left(0(-1)^{1+3}\begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix}\right.$$

$$+\, 0(-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}$$

$$\left. +\, 2(-1)^{3+2}\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix}\right)$$

$$= 4\left(0 + 0 + 2(-1)^{5} * 5\right)$$

$$= 4(-10) = -40$$

Solution of Set linear equations by using inverse matrix method 25

system of equations

$[A][X] = [B]$

$[A^{-1}][A][X] = [A^{-1}][B]$

$[I][X] = [A^{-1}][B]$

$[X] = [A^{-1}][B]$

ex / Solve the system of linear equations using inverse matrix method.

$x - 2y = 1$

$2x + 3y + z = 7$

$-x + 2z = 8$

Solution

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & 3 & 1 \\ -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 8 \end{bmatrix} \Rightarrow [A][X] = [B]$$

1- $\det A = \left|\begin{array}{ccc|cc} 1 & -2 & 0 & 1 & -2 \\ 2 & 3 & 1 & 2 & 3 \\ -1 & 0 & 2 & -1 & 0 \end{array}\right| = 6 + 2 + 0 - 0 - 0 + 8$

$= 16 \neq 0$

2- Cof A

$A_{11} = (+)6 \quad A_{12} = (-)5 \quad A_{13} = (+)3$

$A_{21} = (-1)(-4) \quad A_{22} = (+)2 \quad A_{23} = (-1)(-2)$

$A_{31} = (+)(-2) \quad A_{32} = (-1)1 \quad A_{33} = (+)7$

$$\therefore \text{Cof } A = \begin{bmatrix} 6 & -5 & 3 \\ 4 & 2 & 2 \\ -2 & -1 & 7 \end{bmatrix}$$

3- $\text{adj } A = [\text{Cof } A]^T$

26

$$adj\,A = \begin{bmatrix} 6 & 4 & -2 \\ -5 & 2 & -1 \\ 3 & 2 & 7 \end{bmatrix}$$

4- $A^{-1} = \dfrac{adj\,A}{|A|} = \dfrac{1}{16}\begin{bmatrix} 6 & 4 & -2 \\ -5 & 2 & -1 \\ 3 & 2 & 7 \end{bmatrix}$

$$[X] = [A^{-1}][B] \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16}\begin{bmatrix} 6 & 4 & -2 \\ -5 & 2 & -1 \\ 3 & 2 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 7 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{16}\begin{bmatrix} (6)(1) + (4)(7) + (-2)(8) \\ (-5)(1) + (2)(7) + (-1)(8) \\ (3)(1) + (2)(7) + (7)(8) \end{bmatrix} = \begin{bmatrix} \frac{18}{16} \\ \frac{1}{16} \\ \frac{73}{16} \end{bmatrix}$$

The Solution is

$x = \frac{18}{16}$, $y = \frac{1}{16}$, $z = \frac{73}{16}$

Ex Solve the equations $3x+4y+5z=18$, 27

$2x+y-8z=13$ and $5x-2y+7z=20$

by matrix inversion method

Solution

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix} \Rightarrow [A][x]=[B]$$

1- $\det A = |A| = 3(-7+16)-4(14-40)+5(-4+5)=136$

2- The matrix formed by the cofactors of the elements be

$$D=\begin{bmatrix} (-7+16) & -(14-40) & (-4+5) \\ -(28+10) & (21-25) & -(-6-20) \\ (32+5) & -(24-10) & (-3-8) \end{bmatrix}=\begin{bmatrix} 9 & 26 & 1 \\ -38 & -4 & 26 \\ 37 & -14 & -11 \end{bmatrix}$$

3- $\operatorname{adj} A = D^T = \begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$

4 $A^{-1} = \dfrac{\operatorname{adj} A}{\det A} = \dfrac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}$

5- $[x] = [A^{-1}][B]$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{136}\begin{bmatrix} 9 & -38 & 37 \\ 26 & -4 & -14 \\ 1 & 26 & -11 \end{bmatrix}\begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

The solution $x=3$, $y=1$, $z=1$

ex/ Solve the system of equations by matrix method

$x_1 + x_2 + x_3 = 2$, $4x_1 - x_2 + 2x_3 = -6$, $3x_1 + x_2 + x_3 = -18$

Solution

$[A][x] = [B]$

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ -18 \end{bmatrix}$$

1- $\det A = |A| = 1(-3) - 1(-2) + 7 = 6 \neq 0 \Rightarrow A^{-1}$ exists

2 $\operatorname{cof} A = \begin{bmatrix} -3 & 2 & 7 \\ 0 & -2 & 2 \\ 3 & 2 & -5 \end{bmatrix}$

3- $\operatorname{adj} A = [\operatorname{cof} A]^{T} = \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 7 & 2 & -5 \end{bmatrix}$

4 $A^{-1} = \dfrac{\operatorname{adj} A}{\det A} = \dfrac{1}{6} \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 7 & 2 & -5 \end{bmatrix}$

$[X] = [A^{-1}][B]$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -3 & 0 & 3 \\ 2 & -2 & 2 \\ 7 & 2 & -5 \end{bmatrix} \begin{bmatrix} 2 \\ -6 \\ -18 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -6-54 \\ 4+12-36 \\ 14-12+90 \end{bmatrix} = \begin{bmatrix} -10 \\ \frac{-20}{6} \\ \frac{92}{6} \end{bmatrix}$$

The Solution is

$x_1 = -10$, $x_2 = \dfrac{-10}{3}$, $x_3 = \dfrac{46}{3}$

Ex solve the system of equations by matrix method

$x_1 + 2x_2 + 3x_3 = -1$

$2x_1 + 5x_2 + 7x_3 = -2$

$-2x_1 - 4x_2 - 5x_3 = 0$

Solution

$[A][x] = [B]$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$$

$$\det(A) = |A| = 1 \times (-25 + 28) - 2 \times (-10 + 14) + 3(-8 + 10)$$

$$= 3 - 8 + 6 = 1 \neq 0$$

$$\text{cof } A = \begin{bmatrix} 3 & -4 & 2 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\text{adj } A = \{\text{cof } A\}^T = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{1} \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\{x\} = \{A^{-1}\}\{B\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$$

The Solution is

$x_1 = 1$, $x_2 = 2$, $x_3 = -2$

H.w

30

Solve by using inverse matrix method

$-x + 2y + 3z = 1$

$2x + z = -2$

$4x - 2y + z = 3$

Cramer's Rule (determinant method) 31

The Solution of the system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

is given by $x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$, $\Delta \neq 0$

where

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

We notice that $\Delta_1, \Delta_2, \Delta_3$ are the determinants obtained from Δ on replacing the first, second and third columns by "d"s respectively.

Ex1 Solve the equations $2x+y-z=1$, $x-y+z=2$ 32
$5x+5y-4z=3$ by Cramer's rule.

Solution

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 5 & 5 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\det A = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 5 & 5 & -4 \end{vmatrix} = 2(4-5)-1(-4+5)-1(5+5) = -3$$

$$D_1 = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & 5 & -4 \end{vmatrix} = 1(4-5)-1(-8-3)-1(10+3) = -3$$

$$D_2 = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ 5 & 3 & -4 \end{vmatrix} = 2(-8-3)-1(-4-5)-1(3-10) = -6$$

$$D_3 = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 5 & 5 & 3 \end{vmatrix} \quad \text{or}$$

$$x_1 = \frac{D1}{D} = \frac{-3}{-3} = 1 \quad , y = \frac{D2}{D} = \frac{-6}{-3} = 2$$

Substituting $x=1$, $y=2$ in the equation
$2x+y-z=1$, we get
$2(1)+2-z=1 \Rightarrow z=3$
the Solution is $x=1$, $y=2$, $z=3$

33

Ex / Solve $3x_1 - 2x_2 = 6$, $2x_1 + x_2 = 0.5$

Solution

$[A]\{x\} = [B]$

$$\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0.5 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = (3)(1) - (-2)(2) = 3 + 4 = 7 = D$$

$$D_1 = \begin{vmatrix} 6 & -2 \\ 0.5 & 1 \end{vmatrix} = (6)(1) - (-2)(0.5) = 7$$

$$D_2 = \begin{vmatrix} 3 & 6 \\ 2 & 0.5 \end{vmatrix} = (3)(0.5) - (6)(2) = -10.5$$

$$x_1 = \frac{D1}{D} = \frac{7}{7} = 1$$

$$x_2 = \frac{D2}{D} = \frac{-10.5}{7} = -1.5$$

Solve $x_1 + 2x_2 + x_3 = 0$

$3x_1 - x_2 - 2x_3 = 9$

$4x_1 + 3x_2 - 3x_3 = 3$ 34

Solution

$[A][x] = [B]$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -2 \\ 4 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 3 \end{bmatrix}$$

$$\det(A) = |A| = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -1 & -2 \\ 4 & 3 & -3 \end{vmatrix} = 24 = D$$

$$D1 = \begin{vmatrix} 0 & 2 & 1 \\ 9 & -1 & -2 \\ 3 & 3 & -3 \end{vmatrix} = 72$$

$$x_1 = \frac{D1}{D} = \frac{72}{24} = 3$$

$$D_2 = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 9 & -2 \\ 4 & 3 & -3 \end{vmatrix} = -48$$

$$x_2 = \frac{D_2}{D} = \frac{-48}{24} = -2$$

$$D3 = \begin{vmatrix} 1 & 2 & 0 \\ 3 & -1 & 9 \\ 4 & 3 & 3 \end{vmatrix} = 24$$

$$x_3 = \frac{D3}{D} = \frac{24}{24} = 1$$

Ex / Solve the following system of linear eq. using crammer's rule

$-2x + 3y - z = 1$

$x + 2y - z = 4$

$-2x - y + z = -3$

Solution

$[A][x] = [B]$

$$\begin{bmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2 \neq 0 = D$$

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix}}{-2} = \frac{-4}{-2} = 2$$

$$y = \frac{D_2}{D} = \frac{\begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix}}{-2} = \frac{-6}{-2} = 3$$

$$z = \frac{D_2}{D} = \frac{\begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix}}{-2} = \frac{-8}{-2} = 4$$

the Solution is $x = 2$, $y = 3$, $z = 4$

H.w

36

Solve by using Grammer's rule

$-x + 2y + 3z = 1$

$2x + z = -2$

$4x - 2y + z = 3$

Solve the equations $\quad$ 37

$$x_1 + 2x_2 - x_3 = 2$$
$$3x_1 + 6x_2 + x_3 = 1$$
$$3x_1 + 3x_2 + 2x_3 = 3$$

i by using Cramer's Rule

ii by matrix Inversion

Solution

(i)

$$x_1 = \frac{|B_1|}{|A|}, \quad x_2 = \frac{|B_2|}{|A|}, \quad x_3 = \frac{|B_3|}{|A|}$$

where

$$|B_1| = \begin{vmatrix} 2 & 2 & -1 \\ 1 & 6 & 1 \\ 3 & 3 & 2 \end{vmatrix} = 18 - 7 + 24 = 35$$

$$|B_2| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 3 & 3 & 2 \end{vmatrix} = -6 + 5 - 12 = -13$$

$$|B_3| = \begin{vmatrix} 1 & 2 & 2 \\ 3 & 6 & 1 \\ 3 & 3 & 3 \end{vmatrix} = -18 + 3 + 0 = -15$$

$$|A| = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 6 & 1 \\ 3 & 3 & 2 \end{vmatrix} = 9 - 6 + 9 = 12$$

$$x_1 = \frac{35}{12}, \quad x_2 = \frac{-13}{12}, \quad x_3 = -\frac{15}{12}$$

ii) The cofactors of elements a_{ij} are

$$A_{11} = (-1)^{1+1} \begin{vmatrix} 6 & 1 \\ 3 & 2 \end{vmatrix} = 9, \quad A_{12} = (-1)^{1+2} \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = -3$$

$A_{13} = -9 \quad A_{21} = -7, \; A_{22} = 5$

$A_{23} = 3, \; A_{31} = 8, \; A_{32} = -4, \; A_{33} = 0$

$$\therefore \text{cofactor matrix} = [A_{ij}] = \begin{bmatrix} 9 & -3 & -9 \\ -7 & 5 & 3 \\ 8 & -4 & 0 \end{bmatrix}$$

$$[A_{ij}]^T = adj[A] = \begin{bmatrix} 9 & -7 & 8 \\ -3 & 5 & -4 \\ -9 & 3 & 0 \end{bmatrix}$$

$$[A]^{-1} = \frac{adj[A]}{|A|} = \frac{1}{12} \begin{bmatrix} 9 & -7 & 8 \\ -3 & 5 & -4 \\ -9 & 3 & 0 \end{bmatrix}$$

$$X = [A]^{-1} C = \frac{1}{12} \begin{bmatrix} 9 & -7 & 8 \\ -3 & 5 & -4 \\ -9 & 3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

$$X = \frac{1}{12} \begin{bmatrix} 35 \\ -13 \\ -15 \end{bmatrix}$$

$$X_1 = \frac{35}{12}, \quad X_2 = \frac{-13}{12}, \quad X_3 = \frac{-15}{12}$$

Ex Find A^{-1} To 39

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

Solution

$$|A| = \left| \begin{matrix} 2 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 2 & 2 \end{matrix} \begin{matrix} 2 & 3 \\ 2 & 2 \\ 3 & 2 \end{matrix} \right|$$

$$= 8 + 27 + 8 - 12 - 12 - 12 = 7$$

$$a_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -2$$

$$a_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 5$$

$$a_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = -2$$

$$a_{21} = (-1)^{2+1} \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = -2$$

$$a_{22} = (-1)^{2+2} \begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} = -2$$

$$a_{23} = (-1)^{2+3} \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 5$$

40

$$a_{31} = (-1)^{3+1} \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = 5$$

$$a_{32} = (-1)^{3+2} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} = -2$$

$$A^{-1} = \frac{\operatorname{adj} A}{|A|} = \begin{bmatrix} \frac{a_{11}}{|A|} & \frac{a_{21}}{|A|} & \frac{a_{n1}}{|A|} \\ \frac{a_{12}}{|A|} & \frac{a_{22}}{|A|} & \frac{a_{n2}}{|A|} \\ \frac{a_{1n}}{|A|} & \frac{a_{2n}}{|A|} & \frac{a_{nn}}{|A|} \end{bmatrix}$$

$$a_{33} = (-1)^{3+3} \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -2$$

$$\operatorname{Cof} A = \begin{bmatrix} -2 & 5 & -2 \\ -2 & -2 & 5 \\ 5 & -2 & -2 \end{bmatrix}$$

$$\operatorname{adj} A = [\operatorname{Cof.} A]^{T}$$

$$\operatorname{adj} A = \begin{bmatrix} -2 & -2 & 5 \\ 5 & -2 & -2 \\ -2 & 5 & -2 \end{bmatrix}$$

4)

$$A^{-1}\begin{bmatrix} \frac{-2}{7} & \frac{-2}{7} & \frac{5}{7} \\ \frac{5}{7} & \frac{-2}{7} & \frac{-2}{7} \\ \frac{-2}{7} & \frac{5}{7} & \frac{-2}{7} \end{bmatrix}$$