AL-Karkh University of Science
College of Geophysics and Remote Sensing
Department of Geophysics

Numerical Methods

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2018/2019
For Geophysics students

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College of Remote Sensing and Geophysics
Department of Geophysics
Title of the course **Numerical Methods**
Level : 3rd Stage, 1st Semester
Hours: 2 Hours / Week

References:

1- Mathematical Methods Dr. T. K. V. Iyengar , Dr. B Krishna Gandhi (20080

16

Numerical Methods for Solution of Partial Differential Equations

16.1 GENERAL LINEAR PARTIAL DIFFERENTIAL EQUATIONS

General partial differential equation is of the form

$$A(x, y)\frac{\partial^2 u}{\partial x^2} + B(x, y)\frac{\partial^2 u}{\partial x \partial y} + C(x, y)\frac{\partial^2 u}{\partial y^2} + D(x, y)\frac{\partial u}{\partial x} + E(x, y)\frac{\partial u}{\partial y} + F(x, y)\,u + G(x, y) = 0$$

This equation is called

(*i*) *Elliptic*, if $B^2 - 4AC < 0$

e.g. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{Laplace Equation}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y) \qquad \text{Poisson's Equation}$$

(*ii*) *Parabolic*, if $B^2 - 4AC = 0$ *e.g.* $\dfrac{\partial u}{\partial t} = C^2 \dfrac{\partial^2 u}{\partial x^2}$

One dimensional heat conduction equation.

(*iii*) *Hyperbolic*, if $B^2 - 4AC > 0$ *e.g.* $\dfrac{\partial^2 u}{\partial t^2} = C^2 \dfrac{\partial^2 u}{\partial x^2}$

Example 1. *Determine the type of* $x^2 \dfrac{\partial^2 u}{\partial y^2} + 2xy \dfrac{\partial^2 u}{\partial x \partial y} + y^2 \dfrac{\partial^2 u}{\partial x^2} = 0.$

(*A.M.I.E.T.E., Dec. 2006*)

Solution. Here, $A = x^2,\ B = 2xy,\ c = y^2$.

$B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$. Hence, it is a parabolic equation. **Ans.**

16.2 FINITE-DIFFERENCE APPROXIMATION TO DERIVATIVES

By Taylor formula

$$u(x+h, y) = u(x, y) + \frac{h\,\partial u}{\partial x} + \frac{1}{2!}h^2\frac{\partial^2 u}{\partial x^2} + \frac{1}{3!}h^3\frac{\partial^3 u}{\partial x^3} + \qquad \text{...(1)}$$

$$u(x-h, y) = u(x, y) - h\frac{\partial u}{\partial x} + \frac{1}{2!}h^2\frac{\partial^2 u}{\partial x^2} - \frac{1}{3!}h^3\frac{\partial^3 u}{\partial x^3} + \qquad \text{...(2)}$$

From (1), neglecting h^2 and higher powers of h, we get

$$\frac{\partial u}{\partial x} \approx \frac{u(x+h, y) - u(x, y)}{h} \qquad \text{(Forward difference formula) ...(3)}$$

From (2), neglecting h^2 and higher powers of h, we have

$$\frac{\partial u}{\partial x} \approx \frac{u(x, y) - u(x-h, y)}{h} \qquad \text{(Backward difference formula) ...(4)}$$

Subtracting (2) from (1) and neglecting h^3 and higher power of h we get

$$u(x+h, y) - u(x-h, y) \approx 2h\frac{\partial u}{\partial x}$$

or $$\frac{\partial u}{\partial x} \approx \frac{1}{2h}[u(x+h, y) - u(x-h, y)] \qquad \text{(Central difference formula)} \qquad ...(5)$$

Similarly,

$$\frac{\partial u}{\partial y} = \frac{u(x, y+k) - u(x, y)}{k} = \frac{u(x, y) - u(x, y-k)}{k} = \frac{u(x, y+k) - u(x, y-k)}{2k} \qquad ...(6)$$

Adding (1) and (2), neglecting h^5 and higher powers of h, we get

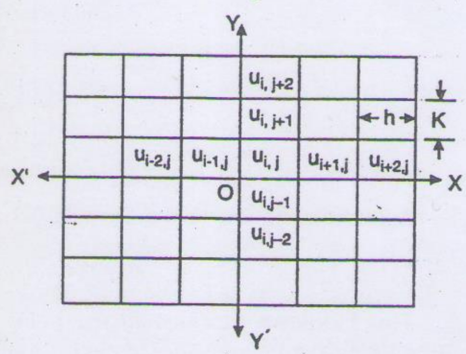
$$u(x+h, y) + u(x-h, y) = 2u(x, y) + h^2 \frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{1}{h^2}[u(x+h, y) - 2u(x, y) + u(x-h, y)] \qquad ...(7)$$

Similarly $$\frac{\partial^2 u}{\partial y^2} \approx \frac{1}{k^2}[u(x, y+k) - 2u(x, y) + u(x, y-k)] \qquad ...(8)$$

and $$\frac{\partial^2 u}{\partial x \partial y} \approx \frac{1}{4hk}[u(x+h, y+k) - u(x-h, y+k) - u(x+h, y-k) + u(x-h, y-k)] \qquad ...(9)$$

The given region (rectangle $ABCD$) is divided into smaller rectangles of sides $\delta x = h$ and $\delta y = k$. The origin is taken at the centre of the rectangle and the coordinates axes are drawn. The rectangle is divided into 36 small rectangles. Here there are 49 mesh-points or lattices or nodal points or grid points. The values of the function u are $u_{i,j}$, $u_{i+1,j}$, $u_{i+2,j}$ ... $u_{i,j+1}$, $u_{i,j+2}$ ..., at the mesh-points.



Let these values satisfy the given partial differential equation.

At the centre of the rectangle:

Equations (5), (6), (7), (8) and (9) are rewritten on the nodal points as below:

$$\frac{\partial u}{\partial x} = \frac{1}{2h}(u_{i+1,j} - u_{i-1,j}), \quad \frac{\partial u}{\partial y} = \frac{1}{2k}(u_{i,j+1} - u_{i\,j-1})$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{k^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1})$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{4hk}(u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1})$$

16.3 SOLUTION OF PARTIAL DIFFERENTIAL EQUATION (LAPLACE EQUATION)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(1)$$

On substituting the values of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$, we get

$$\frac{1}{h^2}[u(x+h, y) - 2u(x, y) + u(x-h, y)] + \frac{1}{k^2}[u(x, y+k) - 2u(x, y) + u(x, y-k)] = 0$$

For values of $h = k$ *i.e.* for square grid of the mesh size h, the above equation can be written as

$$u(x+h, y) - 2u(x, y) + u(x-h, y) + u(x, y+h) - 2u(x, y) + u(x, y-h) = 0$$

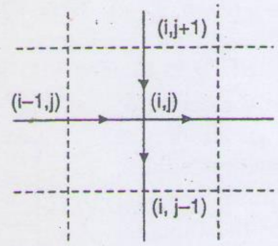
$$u(x, y) = \frac{1}{4}[u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h)]$$

Denoting any mesh point $(x, y) = (ih, jh)$ as simply i, j, the above difference equation can be written as

$$u_{i,j} = \frac{1}{4}(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) \quad ...(2)$$

Equation (2) shows that the value of $u(x, y)$ is the average of its four neighbours to the East, West, North and South. The formula (2) is called the Standard five points formula and is written as

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$$

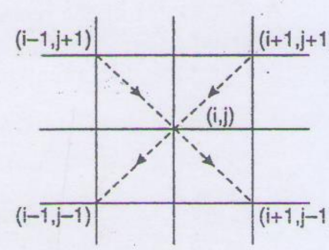


This formula is also known as Liebman's averaging procedure.

A formula similar to the formula (2) is sometimes used with convenience. It is given as

$$u_{i,j} = \frac{1}{4}(u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1}\ u_{i-1,j-1}) \quad ...(3)$$

This is known as *diagonal five-point formula* as these points lie on the diagonals. Although formula (3) is less accurate than formula (2), still it is a good approximation for obtaining as starting values in the iteration procedure.

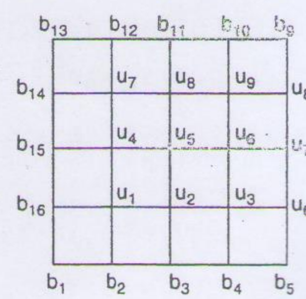


Whenever possible, Standard five-point formula is preferred in all commutations.

Procedure. We use the following diagonal five point formula to get the initial value of u at the centre.

$$u_5 = \frac{1}{4}[b_1 + b_5 + b_9 + b_{13}]$$

Then the approximate values of u_1, u_3, u_7, u_9 are calculated by the diagonal five-point formula



$$u_1 = \frac{1}{4}[b_1 + b_2 + b_5 + b_{15}], \quad u_3 = \frac{1}{4}[b_3 + b_5 + b_7 + u_5]$$

$$u_7 = \frac{1}{4}[b_{15} + u_5 + b_{11} + b_{13}], \quad u_9 = \frac{1}{4}[u_5 + b_7 + b_9 + b_{11}]$$

The values of the remaining interior points *i.e.* u_2, u_4, u_6 and u_8 are obtained by the standard five point formula.

$$u_2 = \frac{1}{4}[b_3 + u_3 + u_5 + u_1], \quad u_4 = \frac{1}{4}[u_1 + u_5 + u_7 + b_{15}]$$

$$u_6 = \frac{1}{4}[u_3 + b_7 + u_9 + u_5], \quad u_8 = \frac{1}{4}[u_5 + u_9 + b_{11} + u_7]$$

Having obtained all values $u_1, u_2, \ldots u_9$ once, their accuracy can be improved by the repeated application of either Jacobi's iteration formula or Gauss-Seidel iteration formula.

16.4 JACOBI'S ITERATION FORMULA

Let $u^n_{i,j}$ be the nth iterative value of $u_{i,j}$ Then Jacobi's iterative procedure is given below.

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n)}_{i,j-1} + u^{(n)}_{i,j+1}] \; .$$

16.5 GAUSS-SEIDEL METHOD

This method utilises the latest iterative value available and scans the mesh points symmetrically from left to right along successive rows. The formula is given below.

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n+1)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n+1)}_{i,j-1} + u^{(n)}_{i,j+1}].$$

16.6 SUCCESSIVE OVER-RELAXATION OR S.O.R. METHOD

Gauss-Seidel formula can be written as

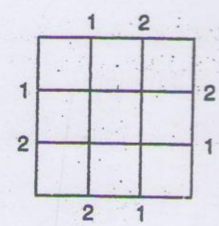
$$u^{(n+1)}_{i,j} = u^{(n)}_{i,j} + \frac{1}{4}[u^{(n+1)}_{i-j} + u^{(n)}_{i+1,j} + u^{(n+1)}_{i,j-1} + u^{(n)}_{i,j+1} - 4u^{(n)}_{i,j}] = u^{(n)}_{i,j} + \frac{1}{4}R_{i,j}.$$

It gives the change $\frac{1}{4}R_{i,j}$ in the value of $u_{i,j}$ for one Gauss-Seidel iteration. In the *S.O.R.* method, larger change than this is given to $u^{(n)}_{i,j}$ and the iteration formula is given below:

$$u^{(n+1)}_{i,j} = u^{(n)}_{i,j} + \frac{1}{4}wR_{i,j} = \frac{1}{4}w[u^{(n+1)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n+1)}_{i,j-1} + u^{(n)}_{i,j+1}] + (1-w)u^{(n)}_{i,j}$$

Here w is called the accelerating factor and lies between 1 and 2.

Example 1. *Solve* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *in the domain of the figure given below by Gauss-Seidel method.*



Solution. Initially $u_1 = u_2 = u_3 = u_4 = 0$

$$u_1^{(n+1)} = \frac{1}{4}\left(1 + 1 + u_2^{(n)} + u_1^{(n)}\right)$$

$$u_2^{(n+1)} = \frac{1}{4}\left(2 + 2 + u_1^{(n+1)} + u_3^{(n)}\right)$$

$$u_3^{(n+1)} = \frac{1}{4}(2 + 2 + u_1^{(n+1)} + u_3^{(n+1)})$$

$$u_4^{(n+1)} = \frac{1}{4}[2 + 2 + u_1^{(n+1)} + u_3^{(n+1)}]$$

First iteration

$$u_1^{(1)} = \frac{1}{4}(1 + 0 + 1 + 0) = 0.5 \qquad u_2^{(1)} = \frac{1}{4}(2 + 0 + 2 + 0.5) = 1.125$$

$$u_3^{(1)} = \frac{1}{4}(1 + 1.125 + 1 + 0) = 0.781 \qquad u_4^{(1)} = \frac{1}{4}(2 + 0.781 + 2 + 0.5) = 1.320$$

Second iteration

$$u_1^{(2)} = \frac{1}{4}[1 + 1 + 1.125 + 1.320] = 1.111 \qquad u_2^{(2)} = \frac{1}{4}[2 + 2 + 1.111 + 0.781] = 1.473$$

$$u_3^{(2)} = \frac{1}{4}[1+1+1.473+1.320] = 1.198 \qquad u_4^{(2)} = \frac{1}{4}[2+2+1.111+1.198] = 1.577$$

Third iteration

$$u_1^{(3)} = \frac{1}{4}[1+1+1.473+1.577] = 1.263 \qquad u_2^{(3)} = \frac{1}{4}[2+2+1.263+1.198] = 1.615$$

$$u_3^{(3)} = \frac{1}{4}[1+1+1.615+1.577] = 1.298 \qquad u_4^{(3)} = \frac{1}{4}[2+2+1.263+1.298] = 1.640$$

Fourth iteration

$$u_1^{(4)} = \frac{1}{4}[1+1+1.615+1.640] = 1.314 \qquad u_2^{(4)} = \frac{1}{4}[2+2+1.314+1.298] = 1.653$$

$$u_3^{(4)} = \frac{1}{4}[1+1+1.653+1.640] = 1.323 \qquad u_4^{(4)} = \frac{1}{4}[2+2+1.314+1.323] = 1.659$$

Fifth iteration

$$u_1^{(5)} = \frac{1}{4}[1+1+1.653+1.659] = 1.328 \qquad u_2^{(5)} = \frac{1}{4}[2+2+1.328+1.323] = 1.663$$

$$u_3^{(5)} = \frac{1}{4}[1+1+1.663+1.659] = 1.331 \qquad u_4^{(5)} = \frac{1}{4}[2+2+1.328+1.331] = 1.665$$

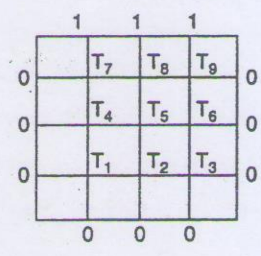
Sixth iteration

$$u_1^{(6)} = \frac{1}{4}[1+1+1.663+1.665] = 1.333 \qquad u_2^{(6)} = \frac{1}{4}[2+2+1.332+1.331] = 1.666$$

$$u_3^{(6)} = \frac{1}{4}[1+1+1.666+1.665] = 1.333 \qquad u_4^{(6)} = \frac{1}{4}[2+2+1.332+1.333] = 1.666$$

$$u_1 = 1.333,\ u_2 = 1.667,\ u_3 = 1.333,\ u_4 = 1.667 \quad \textbf{Ans.}$$

Example 2. *Solve* $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$ *in the domain of the figure given below by Gauss-Seidel method.*



Solution.

$$T_1 = \frac{1}{4}[0+T_2+T_4+0] \qquad T_2 = \frac{1}{4}[0+T_3+T_5+T_1]$$

$$T_3 = \frac{1}{4}[0+0+T_2+T_6] \qquad T_4 = \frac{1}{4}[T_1+T_5+T_7+0]$$

$$T_5 = \frac{1}{4}[T_2+T_6+T_8+T_4] \qquad T_6 = \frac{1}{4}[T_3+0+T_9+T_5],$$

$$T_7 = \frac{1}{4}[T_4+T_8+1+0] \qquad T_8 = \frac{1}{4}[T_5+T_9+1+T_7]$$

$$T_9 = \frac{1}{4}[T_0+0+1+T_8].$$

Gauss-Seidel Method

Initial approximations are

$$T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_7 = T_8 = T_9 = 0$$

Ten successive iterates are given below:

First Iteration

$$T_1^{(n+1)} = \frac{1}{4}[0 + T_4^{(n)} + T_2^{(n)} + 0], \qquad T_1^{(1)} = \frac{1}{4}[0+0+0+0] = 0$$

$$T_2^{(n+1)} = \frac{1}{4}[T_1^{(n+1)} + T_5^{(n)} + T_3^{(n)} + 0], \qquad T_2^{(1)} = \frac{1}{4}[0+0+0+0] = 0$$

$$T_3^{(n+1)} = \frac{1}{4}[T_2^{(n+1)} + T_6^{(n)} + 0 + 0], \qquad T_3^{(1)} = \frac{1}{4}[0+0+0+0] = 0$$

$$T_4^{(n+1)} = \frac{1}{4}[0 + T_7^{(n)} + T_5^{(n)} + T_1^{(n+1)}], \qquad T_4^{(1)} = \frac{1}{4}[0+0+0+0] = 0$$

$$T_5^{(n+1)} = \frac{1}{4}[T_4^{(n+1)} + T_8^{(n)} + T_6^{(n)} + T_2^{(n+1)}], \qquad T_5^{(1)} = \frac{1}{4}[0+0+0+0] = 0$$

$$T_6^{(n+1)} = \frac{1}{4}[T_5^{(n+1)} + T_9^{(n)} + 0 + T_3^{(n+1)}], \qquad T_6^{(1)} = \frac{1}{4}[0+0+0+0] = 0$$

$$T_7^{(n+1)} = \frac{1}{4}[0 + 1 + T_8^{(n)} + T_4^{(n+1)}], \qquad T_7^{(1)} = \frac{1}{4}[0+1+0+0] = 0.25$$

$$T_8^{(n+1)} = \frac{1}{4}[T_7^{(n+1)} + 1 + T_9^{(n)} + T_5^{(n+1)}], \qquad T_8^{(1)} = \frac{1}{4}[0.25+1+0+0] = 0.312$$

$$T_9^{(n+1)} = \frac{1}{4}[T_8^{(n+1)} + 1 + 0 + T_6^{(n+1)}], \qquad T_9^{(1)} = \frac{1}{4}[0.312+1+0+0] = 0.328$$

Second Iteration

$$T_1^{(2)} = \frac{1}{4}[0+0+0+0] = 0 \qquad T_2^{(2)} = \frac{1}{4}[0+0+0+0] = 0$$

$$T_3^{(2)} = \frac{1}{4}[0+0+0+0] = 0 \qquad T_4^{(2)} = \frac{1}{4}[0+0.25+0+0] = 0.062$$

$$T_5^{(2)} = \frac{1}{4}[0+0.312+0+0] = 0.078 \qquad T_6^{(2)} = \frac{1}{4}[0+0.328+0+0] = 0.082$$

$$T_7^{(2)} = \frac{1}{4}[0+1+0.312+0] = 0.328 \qquad T_8^{(2)} = \frac{1}{4}[0.25+1+0.328+0] = 0.394$$

$$T_9^{(2)} = \frac{1}{4}[0.312+1+0+0] = 0.328$$

Third Iteration

$$T_1^{(3)} = \frac{1}{4}[0+0.062+0+0] = 0.016 \qquad T_2^{(3)} = \frac{1}{4}[0.016+0.078+0+0] = 0.024$$

$$T_3^{(3)} = \frac{1}{4}[0.024+0.082+0+0] = 0.027 \qquad T_4^{(3)} = \frac{1}{4}[0+0.328+0.078+ \; +0.016] = 0.106$$

$$T_5^{(3)} = \frac{1}{4}[0.106+0.394+0.082+0.024] = 0.152$$

$$T_6^{(3)} = \frac{1}{4}[0.152+0.328+0+0.027] = 0.127$$

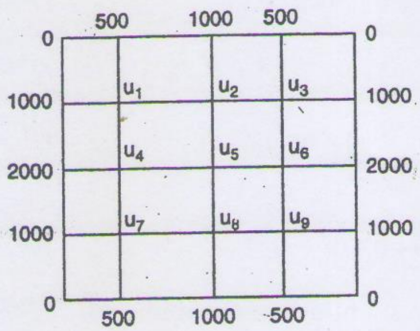
$$T_7^{(3)} = \frac{1}{4}[0+1+0.394+0.106] = 0.375 \qquad T_8^{(3)} = \frac{1}{4}[0.375+1+0.328+0.152] = 0.464$$

$$T_9^{(3)} = \frac{1}{4}[0.464+1+0+0.127] = 0.398$$

and so on. **Ans.**

Iteration	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9
4	0.032	0.053	0.045	0.140	0.196	0.160	0.401	0.499	0.415
5	0.048	0.072	0.058	0.161	0.223	0.174	0.415	0.513	0.422
6	0.058	0.085	0.065	0.174	0.236	0.181	0.422	0.520	0.425
7	0.065	0.092	0.068	0.181	0.244	0.184	0.425	0.524	0.427
8	0.068	0.095	0.070	0.184	0.247	0.186	0.427	0.525	0.428
9	0.070	0.097	0.071	0.186	0.249	0.187	0.428	0.526	0.428
10	0.071	0.098	0.071	0.187	0.250	0.187	0.428	0.526	0.428

Example 3. *Solve* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *by Leibman's iteration process for the domain of the figure given below:*



Solution. Values given on the figure are symmetrical about middle line.

$\therefore$ $$u_1 = u_2 = u_3 = u_7$$

$$u_2 = u_8,\ u_4 = u_6$$

$$u_5 = \frac{1}{4}(2000 + 2000 + 1000 + 1000) = 1500 \quad \text{(Standard formula)}$$

$$u_1 = \frac{1}{4}[0 + 1000 + 1500 + 2000] = 1125 \quad \text{(Diag. formula)}$$

Similarly $$u_1 = u_3 = u_5 = u_7 = 1125$$

$$u_2 = \frac{1}{4}(1000 + 1125 + 1500 + 1125) \approx 1188 \text{ (Standard formula)}$$

Similarly $$u_8 = u_2 = 1188$$

$$u_4 = \frac{1}{4}[1125 + 2000 + 1125 + 1500] \approx 1438 \text{ (Standard formula)}$$

Similarly $$u_4 = u_6 = 1438$$

So $u_1 = 1125$, $u_2 = 1188$, $u_3 = 1125$, $u_4 = 1438$, $u_5 = 1500$, $u_6 = 1438$, $u_7 = 1125$, $u_8 = 1188$, $u_9 = 1125$

Gauss-Seidel Method:

$$u^{(n+1)}_{i,j} = \frac{1}{4}[u^{(n+1)}_{i-1,j} + u^{(n)}_{i+1,j} + u^{(n+1)}_{i,j-1} + u^{(n)}_{i,j+1}].$$

$$u_1^{(n+1)} = \frac{1}{4}[1000 + u_2^{(n)} + 500 + u_4^{(n)}] = u_3^{(n+1)} = u_9^{(n+1)} = u_7^{(n+1)}$$

$$u_2^{(n+1)} = \frac{1}{4}[u_1^{(n+1)} + u_3^{(n+1)} + 1000 + u_5^{(n+1)}] = u_8^{(n+1)}$$

$$u_4^{(n+1)} = \frac{1}{4}[200\,0 + u_5^{(n)} + u_1^{(n+1)} + u_7^{(n+1)}] = u_6^{(n+1)}$$

$$u_5^{(n+1)} = \frac{1}{4}[u_4^{(n+1)} + u_6^{(n+1)} + u_2^{(n+1)} + u_8^{(n+1)}]$$

First Iteration

$$u_1^{(1)} = \frac{1}{4}[1000 + 1188 + 500 + 1438] \approx 1032 = u_3^{(1)} = u_9^{(1)} = u_7^{(1)}$$

$$u_2^{(1)} = \frac{1}{4}[1032 + 1032 + 1000 + 1500] = 1141 = u_8^{(1)}$$

$$u_4^{(n+1)} = \frac{1}{4}[2000 + 1500 + 1032 + 1032] = 1391 = u_6^{(1)}$$

$$u_5^{(n+1)} = \frac{1}{4}[1091 + 1391 + 1141 + 1141] = 1266$$

Second Iteration

$$u_1^{(2)} = \frac{1}{4}[1000 + 1141 + 500 + 1391] = 1008 = u_3^{(2)} = u_9^{(2)} = u_7^{(2)}$$

$$u_2^{(2)} = \frac{1}{4}[1008 + 1008 + 1000 + 1266] = 1069 = u_8^{(2)}$$

$$u_4^{(2)} = \frac{1}{4}[2000 + 1266 + 1008 + 1008] = 1321 = u_6^{(2)}$$

$$u_5^{(2)} = \frac{1}{4}[1321 + 1321 + 1069 + 1069] = 1195$$

Similarly

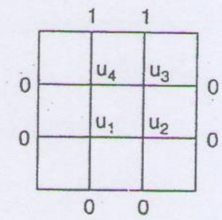
Iteration	$u_1 = u_3 = u_9 = u_7$	$u_2 = u_8$	$u_4 = u_6$	u_5
Third	973	1035	1288	1162
Fourth	956	1019	1269	1144
Fifth	947	1010	1260	1135
Sixth	942	1005	1255	1130
Seventh	940	1003	1253	1128
Eighth	939	1002	1252	1127
Ninth	939	1001	1251	1126

Very small difference is in the eighth and ninth iteration

Thus,
$$u_1 = u_3 = u_7 = u_9 = 939$$
$$u_2 = u_8 = 1001,$$
$$u_4 = u_6 = 1251,$$
$$u_5 = 1126$$ **Ans.**

Example 4. *Solve* $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ *in the domain of the figure given below by*

(*a*) *Jacobi's method*, (*b*) *Gauss-Seidel method and*
(*c*) *Successive Over-Relaxation method*



Solution. **(*a*) Jacobi's Method**

$$u_1^{(1)} = \frac{1}{4}[0+0+0+1] = 0.25$$

$$u_2^{(2)} = \frac{1}{4}[0+0+0+1] = 0.25$$

$$u_3^{(1)} = \frac{1}{4}[1+1+0+0] = 0.5$$

$$u_4^{(1)} = \frac{1}{4}[1+1+0+0] = 0.5$$

Seven successive iterates are given below:

u_1	u_2	u_3	u_4
0.1875	0.1875	0.4375	0.4375
0.15625	0.15625	0.40625	0.40625
0.14062	0.14062	0.39062	0.39062
0.13281	0.13281	0.38281	0.38281
0.12891	0.12891	0.37891	0.37891
0.12695	0.12695	0.37695	0.37695
0.12598	0.12598	0.37598	0.37518

(*b*) Gauss-Seidel Method

Five successive iterates are given below:

u_1	u_2	u_3	u_4
0.25	0.3125	0.5625	0.46875
0.21875	0.17187	0.42187	0.39844
0.14844	0.13672	0.38672	0.38086
0.13086	0.12793	0.37793	0.37646
0.12646	0.12573	0.37573	0.37537

(*c*) Successive Over-Relaxation method

Three successive iterates are given below:

u_1	u_2	u_3	u_4
0.275	0.35062	0.35062	0.35062
0.16534	0.10683	0.38183	0.37432
0.11785	0.12181	0.37216	0.37341

16.7 POISSON EQUATION

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

In this case the standard five-point formula is of the form

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4\,u_{i,j} = h^2 f(ih, jh).$$

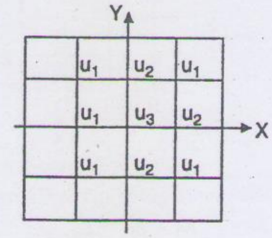
On applying the above formula we get equations. These equations can be solved by Gauss-Seidel iteration method.

Example 5. *Solve the Poisson's equation* $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = 8\,x^2y^2$ *for the square mesh of the figure given below with* $u\,(x, y) = 0$ *on the boundary and mesh length = 1.*

Solution. Here $h = 1$

The standard five-point formula for the given equation is

$$u_{i-1,j}+u_{i+1,j}+u_{i,j+1}+u_{i,j-1}-4\,u_{i,j} = 8\,i^2j^2 \qquad ...(1)$$



For $u_1\,(i = -1, j = +1)$, equation (1) becomes

$$0+u_2+0+u_2-4\,u_1 = 8\,(-1)^2\,(1)^2 \quad \text{or} \quad 4\,u_1 = 2\,u_2-8 \qquad ...(2)$$

For $u_2\,(i = 0, j = 1)$, equation (1) becomes

$$u_1+u_1+0+u_3-4\,u_2 = 0 \quad \text{or} \quad 4\,u_2 = 2\,u_1+u_3 \qquad ...(3)$$

For $u_3\,(i = 0, j = 0)$, equation (1) becomes

$$u_2+u_2+u_2+u_2-4\,u_3 = 0 \quad \text{or} \quad 4\,u_3 = 4\,u_2 \quad \text{or} \quad u_3 = u_2 \qquad ...(4)$$

Putting u_2 for u_3 in (3), we get

$$4\,u_2 = 2\,u_1+u_2 \quad \text{or} \quad 3\,u_2 = 2\,u_1$$

Putting $\dfrac{2\,u_1}{3}$ for u_2 in (2), we get

$$4\,u_1 = \frac{4\,u_1}{3}-8 \quad \text{or} \quad 12\,u_1 = 4\,u_1-24$$

$$8\,u_1 = -24 \quad \text{or} \quad u_1 = -3$$

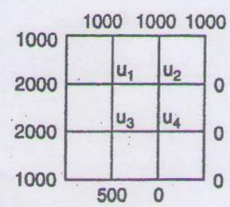
$$u_2 = \frac{2\,u_1}{3} = \left(\frac{2}{3}\times -3\right) = -2, \quad u_3 = u_2 = -2$$

$$u_1 = -3, u_2 = -2, u_3 = -2 \qquad \textbf{Ans.}$$

Exercise 16.1

1. Given the values of $u\,(x, y)$ on the boundary of the square in the figure given below, evaluate the function $u\,(x, y)$ satisfying the Laplace equation

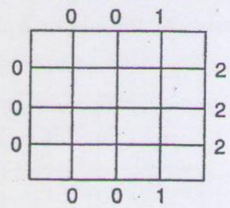
$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0$$



at pivotal points of this figure. **Ans.** $u_1 = 1208, u_2 = 792, u_3 = 1042, u_4 = 458.$

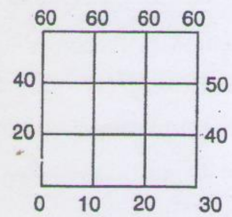
2. Solve $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = 0$

for the square mesh with boundary values as shown in the figure given below. Iterate until the maximum difference between successive values at any point is less than 0.005.



Ans. $u_1 = 10.188,\ u_2 = 0.5, u_3 = 1.188, u_4 = 0.25, u_5 = 0.625, u_6 = 1.25.$

3. Solve $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial t^2} = 0$ within the square given in the figure below.



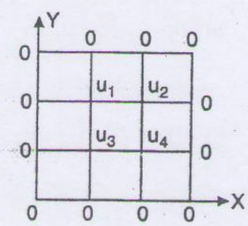
Ans. $u_1 = 26.66, u_2 = 33.33, u_3 = 43.33, u_4 = 46.66.$

4. Solve $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = -10(x^2+y^2+10)$

over the square with $x = 0 = y, x = 3 = y$ with

$u = 0$ on the boundary and mess length = 1.

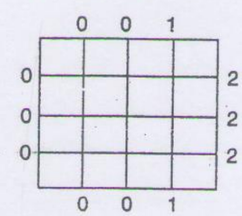
Ans. $u_1 = 75, u_2 = 82.5, u_3 = 67.05, u_4 = 75.$



5. Solve $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2} = 0$ with boundary values as shown in the figure given below.

Ans. $u_1 = 10.188,\ u_2 = 0.5,\ u_3 = 1.188$

$u_4 = 0.25, u_5 = 0.625, u_6 = 1.25$



16.8 HEAT EQUATION (PARABOLIC EQUATION)

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{...(1)}$$

We know that
$$\frac{\partial u}{\partial t} = \frac{u(x, t+k) - u(x, t)}{k}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2}$$

On putting the values of $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial^2 u}{\partial x^2}$ in (1), we get

$$\frac{u(x, t+k) - u(x, t)}{k} = c^2 \frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2}$$

$$u(x, t+k) = \frac{c^2 k}{h^2} u(x+h, t) - \frac{2c^2 k}{h^2} u(x, t) + u(x, t) + \frac{c^2 k}{h^2} u(x-h, t)$$

or
$$u(x, t+k) = au(x+h, t) + (1-2a) u(x, t) + au(x-h, t)$$

If $a = \frac{1}{2}$

$$u(x, t+k) = \frac{1}{2} u(x+h, t) + \frac{1}{2} u(x-h, t) \quad \text{or} \quad u_{i,j+1} = \frac{1}{2}[u_{i+1,j} + u_{i-1,j}]$$

It means that the value of u at x_i at time t is the mean of the values of u at x_{i-1} and x_{i+1} at the previous time t_j.

This relation is known as Bendre-Schimidt recurrence relation.

Example 6. *Find the values of $u(x, t)$ satisfying the parabolic equation $\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(0, t) = 0 = u(8, t)$ and $u(x, 0) = 4x - \frac{1}{2}x^2$ at the points.*

$$x = i : i = 0, 1, 2, 3, \ldots, 7 \text{ and } t = \frac{1}{8}j : j = 0, 1, 2, 3, \ldots, 5.$$

Solution. $c^2 = 4$, $h = 1$, $k = \frac{1}{8}$. $\quad a = \frac{c^2 k}{h^2} = \frac{4 \times 1/8}{(1)} = \frac{1}{2}$

Then the equation is

$$u_{i,j+1} = \frac{1}{2}(u_{i-1,j} + u_{i+1,j}) \qquad \ldots(1)$$

Given $u(0, t) = 0 = u(8, t)$

or $\quad u(0, j) = 0 = u(8, j)$ for all values of $j = 1, 2, 3, 4, 5$

and $\quad u(x, 0) = 4x - \frac{1}{2}x^2 \qquad u_{i,0} = 4i - \frac{1}{2}i^2$

$u_{0,0} = 0$, $\quad u_{1,0} = 4(1) - \frac{1}{2}(1)^2 = 3.5$, $\quad u_{2,0} = 4(2) - \frac{1}{2}(2)^2 = 6$

$u_{3,0} = 7.5$, $u_{4,0} = 8$, $u_{5,0} = 7.5$, $u_{6,0} = 6$, $u_{7,0} = 3.5$.

These entries are shown in the following table :

j \ i	0	1	2	3	4	5	6	7	8
0	0	3.5	6	7.5	8	7.5	6	3.5	0
1	0	3	5.5	7	7.5	7	5.5	3	0
2	0	2.75	5	6.5	7	6.5	5	2.75	0
3	0	2.5	4.625	6	6.5	6	4.625	2.5	0
4	0	2.3125	4.25	5.5625	6	5.5625	4.25	2.3125	0
5	0	2.125	3.9375	5.125	5.5625	5.125	3.9375	2.125	0

Putting $j = 0$ in (1) we have

$$u_{i,1} = \frac{1}{2}(u_{i-1,0} + u_{i+1,0})$$

$$u_{1,1} = \frac{1}{2}(u_{0,0} + u_{2,0}) = \frac{1}{2}(0+6) = 3$$

$$u_{2,1} = \frac{1}{2}(u_{1,0} + u_{3,0}) = \frac{1}{2}(3.5 + 7.5) = 5.5$$

$$u_{3,1} = \frac{1}{2}(u_{2,0} + u_{4,0}) = \frac{1}{2}(6 + 8) = 7$$

$$u_{4,1} = \frac{1}{2}(u_{3,0} + u_{5,0}) = \frac{1}{2}(7.5 + 7.5) = 7.5$$

$$u_{5,1} = \frac{1}{2}(u_{4,0} + u_{6,0}) = \frac{1}{2}(8 + 6) = 7$$

$$u_{6,1} = \frac{1}{2}(u_{5,0} + u_{7,0}) = \frac{1}{2}(7.5 + 3.5) = 5.5$$

$$u_{7,1} = \frac{1}{2}(u_{6,0} + u_{8,0}) = \frac{1}{2}(6 + 0) = 3$$

Putting $j = 1$ in (1), we have

$$u_{1,2} = \frac{1}{2}(u_{i-1,1} + u_{i+1,1})$$

$$u_{1,2} = \frac{1}{2}(u_{0,1} + u_{2,0}) = \frac{1}{2}(0 + 5.5) = 2.75$$

$$u_{2,2} = \frac{1}{2}(u_{1,1} + u_{3,1}) = \frac{1}{2}(3 + 7) = 5$$

$$u_{3,2} = 6.5, \quad u_{4,2} = 7, \quad u_{5,2} = 6.5, \quad u_{6,2} = 5, \quad u_{7,2} = 2.75$$

Putting $j = 2$ in (1), we have

$$u_{i,3} = \frac{1}{2}(u_{i-1,2} + u_{i+1,2})$$

$$u_{1,3} = \frac{1}{2}(u_{0,2} + u_{2,2}) = \frac{1}{2}(0 + 5) = 2.5$$

$$u_{2,3} = \frac{1}{2}(u_{1,2} + u_{3,2}) = \frac{1}{2}(2.75 + 6.5) = 4.625$$

$$u_{3,3} = 6, \; u_{4,3} = 6.5, \; u_{5,3} = 6, \; u_{6,3} = 4.625, \; u_{7,3} = 2.5$$

Putting $j = 3$, in (1) we have

$$u_{i,4} = \frac{1}{2}(u_{i-1,3} + u_{i+1,3})$$

$$u_{1,4} = \frac{1}{2}(u_{0,3} + u_{2,3}) = \frac{1}{2}(0 + 4.625) = 2.3125$$

$$u_{2,4} = \frac{1}{2}(u_{1,3} + u_{3,2}) = (2.5 + 6) = 4.25$$

$$u_{3,4} = 0.5625, \quad u_{4,4} = 6, u_{5,4} = 5.5625, \quad u_{6,4} = 4.25, \quad u_{7,4} = 2.3125$$

Putting $j = 4$, in (1), we have

$$u_{i,5} = \frac{1}{2}(u_{i-1,4} + u_{i+1,4})$$

$$u_{1,5} = \frac{1}{2}(u_{0,4} + u_{2,4}) = \frac{1}{2}(0 + 4.25) = 2.125$$

$$u_{2,5} = \frac{1}{2}(u_{1,4} + u_{3,4}) = \frac{1}{2}(2.125 + 5.5625) = 3.9375$$

$$u_{3,5} = 5.125, u_{4,5} = 5.625, u_{5,5} = 5.125, u_{6,5} = 3.9375, u_{7,5} = 2.125$$

16.9 WAVE EQUATION (HYPERBOLIC EQUATION)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

We know that

$$\frac{\partial^2 u}{\partial t^2} = \frac{u(x, t+k) - 2u(x,t) + u(x, t-k)}{k^2} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = \frac{u(x+h, t) - 2u(x,t) + u(x-h, t)}{h^2}$$

Putting the values of $\frac{\partial^2 u}{\partial t^2}$ and $\frac{\partial^2 u}{\partial x^2}$ in (1) we have

$$\frac{u(x, t+k) - 2u(x,t) + u(x, t-k)}{k^2} = c^2 \frac{u(x+h, t) - 2u(x,t) + u(x-h, t)}{h^2}$$

or $$u(x, t+k) - 2u(x,t) + u(x, t-k) = \frac{c^2 k^2}{h^2}[u(x+h, t) - 2u(x,t) + u(x-h, t)]$$

or $$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = a^2c^2[u_{i+1,j} - 2u_{i,j} + u_{i-1,j}] \qquad \left(a = \frac{k}{h}\right)$$

or $$u_{i,j+1} = 2(1 - a^2c^2)u_{i,j} + a^2c^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \qquad ...(2)$$

If $a^2c^2 = 1$, Equation (2) reduces to

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad ...(3)$$

Example 7. *Solve* $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ *with conditions* $u(0, t) = u(1, t) = 0$, $u(x, 0) = 1/2\, x(1-x)$ *and* $u(x, 0) = 0$, *taking* $h = k = 0.1$ *for* $0 \le t \le 0.4$. *Compare your solution with the exact solution* $x = 0.5$ *and* $t = 0.3$.

Solution. $c^2 = 1$. The difference equation for the given equation is

$$u_{i,j+1} = 2(1 - \alpha^2)u_{i,j} + \alpha^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \qquad ...(1)$$

where $\alpha = \frac{k}{h}$. But $\alpha = \frac{0.1}{0.1} = 1$

Equation (1) reduces to

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \qquad ...(2)$$

$$u(0,t) = u(1,t) = 0; \quad u_{0,j} = 0 \quad \text{and} \quad u_{10,j} = 0$$

i.e., the entries in the first column are zero.

since $u(x, 0) = \frac{1}{2}x(1-x)$ $\qquad u(i, 0) = \frac{1}{2}i(1-i)$

$= 0.045, 0.08, 0.105, 0.120, 0.125, 0.120, 0.105$ for

$i = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7$ at $t = 0$

These are the entries of the fist row.

Finally since $u_x(x, 0) = 0$

$$\therefore \qquad \frac{u_{i,j+1} - u_{i,j}}{k} = 0 \quad \text{for } j = 0\ (t = 0), \quad u_{i,1} = u_{i,9}$$

Putting $j = 0$ in equation (2), we get

$$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$$

$$= u_{i-1,0} + u_{i+1,0} - u_{i,1} \qquad (u_{i,1} = u_{i,-1})$$

$$2\,u_{i,1} = u_{i-1,0} + u_{i+1,0}, \quad u_{i,1} = \frac{1}{2}[u_{i-1,0} + u_{i+1,0}]$$

For $i = 1$, $\quad u_{1,1} = \frac{1}{2}[u_{0,0} + u_{2,0}] = \frac{1}{2}[0 + .080] = 0.040$

For $i = 2$, $\quad u_{2,1} = \frac{1}{2}[u_{1,0} + u_{3,0}] = \frac{1}{2}[0.045 + 0.105] = 0.075$

For $i = 3$, $\quad u_{3,1} = [u_{2,0} + u_{4,0}] = \frac{1}{2}[0.08 + 0.120] = 0.100$

For $i = 4$, $\quad u_{4,1} = \frac{1}{2}[u_{3,0} + 4_{5,0}] = \frac{1}{2}[0.105\ 0.125] = 0.115$

For $i = 5$, $\quad u_{5,1} = \frac{1}{2}[u_{4,0} + u_{6,0}] = \frac{1}{2}[0.120 + 0.120] = 0.120$

For $i = 6$, $\quad u_{6,1} = \frac{1}{2}[u_{5,0} + u_{7,0}] = \frac{1}{2}[0.125 + 0.105] = 0.115$

Putting $j = 1$ in equation (2), we get

$$u_{i,2} = u_{i-1,1} + u_{i+1}, - u_{i,0}$$

For $i = 1$, $\quad u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 0.075 - 0.045 = 0.03$

For $i = 2$, $\quad u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 0.040 + 0.100 - 0.08 = 0.060$

For $i = 3$, $\quad u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 0.075 + 0.115 - 0.105 = 0.085$

For $i = 4$, $\quad u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 0.100 + 0.120 - 0.120 = 0.100$

For $i = 5$, $\quad u_{5,2} = u_{4,1} + u_{6,1} = u_{5,0} = 0.115 + 0.115 - 0.125 = 0.105$

Similarly for $\quad j = 2$,

$$u_{i,3} = u_{i-1,2} + u_{i+1,2} - u_{i,1}$$

$$u_{1,3} = 0.020,\ u_{2,3} = 0.040,\ u_{3,3} = 0.060,\ u_{4,3} = 0.075,\ u_{5,3} = 0.80$$

For $j = 3$, $\quad u_{i,4} = u_{i-1,3} + u_{i+1,3} - u_{i,2}$

$$u_{1,4} = 0.010,\ u_{2,4} = 0.02,$$

$$u_{3,4} = 0.030,\ u_{4,4} = 0.040,\ u_{5,4} = 0.048$$

		0	0.1	0.2	0.3	0.4	0.5	0.6
	i / j	0	1	2	3	4	5	6
0	0	0	0.045	0.080	0.105	0.120	0.125	0.120
0.1	1	0	0.040	0.075	0.100	0.115	0.120	0.115
0.2	2	0	0.030	0.060	0.085	0.100	0.105	
0.3	3	0	0.020	0.040	0.060	0.075	0.080	
0.4	4	0	0.010	0.020	0.030	0.040	0.048	

The analytical (exact) solution of the given equation is

$$u = \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}(1 - \cos n\pi)\sin n\pi x \cos n\pi t$$

Comparison of two solutions is given below:

$i = 0.3$	$x =$	0.1	0.2	0.3	0.4	0.5
Numerical solution	$u =$	0.02	0.04	0.06	0.075	0.08
Exact solution	$u =$	0.02	0.04	0.06	0.075	0.08

Ans.

EXERCISE 16.2

1. Solve $\dfrac{\partial^2 u}{\partial x^2} = 2\dfrac{\partial u}{\partial t}$

under conditions $u(0, t) = u(4, t) = 0$ and $u(x, 0) = x(4 - x)$, taking $h = 1$, find the values upto $t = 5$.

Ans. $u_{1,0} = 3,\ u_{2,0} = 4,\ u_{3,0} = 3;\quad u_{1,1} = 2,\ u_{2,1} = 3,\ u_{3,1} = 2$

$u_{1,2} = 1.5,\ u_{2,2} = 2,\ u_{3,2} = 1.5;\quad u_{1,3} = 1,\ u_{2,3} = 1.5,\ u_{3,3} = 1$

$u_{1,4} = 0.75,\ u_{2,4} = 1,\ u_{3,4} = 0.75;\quad u_{1,5} = 0.5,\ u_{2,5} = 0.75,\ u_{3,5} = 0.50$

2. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2};\ 0 \le x \le 1, t \ge 0$ under the conditions that

$u(0, t) = u(1, t) = 0$ and $u(x, 0) = 2x$ for $0 \le x \le \frac{1}{2}$ $= (1 - x)$ for $\frac{1}{2} \le x \le 1.$

Ans. $u_1 = 0.1989, u_2 = 0.3956, u_3 = 0.5834, u_4 = 0.7381, u_6 = 0.7591$

3. Solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2},\ 0 \le x \le 1, t \ge 0.$ under the conditions that

$$\left.\begin{array}{l} u = 0, \text{ at } x = 0 \\ u = 0, \text{ at } x = 1 \end{array}\right] t \ge 0$$

$$u = \sin \pi x \text{ at } t = 0,\ 0 \le x \le 1.$$

find u for $x = 0.8$ at $t = 1$.

Ans. 0.4853

4. Solve $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t}$ under the conditions that $u(0, t) = u(5, t) = 0;\ u(x, 0) = x^2(25 - x^2)$

taking $h = 1$ and $k = \dfrac{1}{2}$

Ans. $u_{1,0} = 24,\ u_{2,0} = 84,\ u_{3,0} = 144,\ u_{4,0} = 144$

$u_{1,1} = 42,\ u_{2,1} = 78,\ u_{3,1} = 78,\ u_{4,1} = 57$

$u_{1,2} = 39,\ u_{2,2} = 60,\ u_{3,2} = 67.5,\ u_{4,2} = 39$

$u_{1,3} = 30,\ u_{2,3} = 53.25,\ u_{3,3} = 49.5,\ u_{4,3} = 33.75$

$u_{1,4} = 26.625,\ u_{2,4} = 39.75,\ u_{3,4} = 43.5,\ u_{4,4} = 24.75$

$u_{1,5} = 19.875,\ u_{2,5} = 35.06,\ u_{3,5} = 32.25,\ u_{4,5} = 21.75$

5. Solve $16\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 u}{\partial t^2}$ taking $h = 1$, upto $t = 1.25$, under the conditions

$u(0, t) = u(5, t) = 0, u_t(x, 0) = 0$ and $u(x, 0) = x^2(5 - x)$.

Ans. $u_{1,0} = 4,\ u_{2,0} = 12,\ u_{3,0} = 18,\ u_{4,0} = 16$

$u_{1,1} = 4,\ u_{2,1} = 12,\ u_{3,1} = 18,\ u_{4,1} = 16$

$u_{1,2} = 8,\ u_{2,2} = 10,\ u_{3,2} = 10,\ u_{4,2} = 2$

$u_{1,3} = 6,\ u_{2,3} = 6,\ u_{3,3} = -6,\ u_{4,3} = -6$

$u_{1,4} = -2,\ u_{2,4} = -10,\ u_{3,4} = -10,\ u_{4,4} = -8$

$u_{1,5} = -16,\ u_{2,5} = -18,\ u_{3,5} = -12,\ u_{4,5} = -4$