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Numerical Methods

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References:

1- Numerical Methods P. kandasamy , K.thilagavathy , K. gunavathy 2008

4

Solution of Simultaneous Linear Algebraic Equations

DIRECT METHODS

4·1 Introduction

We come across, very often, simultaneous linear algebraic equations for its solutions, especially, in the fields of science and engineering. In lower classes, we have solved such equations by Cramer's rule (determinant methods) or by matrix methods. These methods become tedious when the number of unknowns in the system is large. After the availability of computers, we go to numerical methods which are suited for computer operations. These numerical methods are of two types namely : (*i*) direct (*ii*) iterative.

We will study a few methods below:

Solution of Simultaneous Linear Algebraic Equations

4·2 Gauss-Elimination Method (Direct method). This is a *direct* method based on the elimination of the unknowns by combining equations such that the n equations in n unknowns are reduced to an equivalent upper triangular system which could be solved by back substitution.

Consider the n linear equations in n unknowns, *viz.*

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned} \qquad \ldots(1)$$

where a_{ij} and b_i are known constants and x_i's are unknowns.

The system (1) is equivalent to

$$AX = B \qquad \ldots(2)$$

where $A = \begin{pmatrix} a_{11}\, a_{12} \cdots a_{1n} \\ a_{21}\, a_{22} \cdots a_{2n} \\ \cdots\cdots\cdots \\ a_{n1}\, a_{n2} \cdots a_{nn} \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

Now our aim is to reduce the augmented matrix (A, B) to upper triangular matrix.

$$(A, B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right) \qquad \text{...(3)}$$

Now, multiply the first row of (3) (if $a_{11} \neq 0$) by $-\dfrac{a_{i1}}{a_{11}}$ and add to the ith row of (A, B), where $i = 2, 3, \ldots, n$. By this, all elements in the first column of (A, B) except a_{11} are made to zero. Now (3) is of the form

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ 0 & b_{22} & \cdots & b_{2n} & c_2 \\ \cdot & \cdot & \cdots & \cdots & \cdot \\ 0 & b_{n2} & \cdots & b_{nn} & c_n \end{array}\right) \qquad \text{...(4)}$$

Now take the pivot b_{22}. Now, considering b_{22} as the pivot, we will make all elements below b_{22} in the second column of (4) as zeros. That is, multiply second row of (4) by $-\dfrac{b_{i2}}{b_{22}}$ and add to the corresponding elements of the ith row $(i = 3, 4, \ldots, n)$. Now all elements below b_{22} are reduced to zero. Now (4) reduces to

$$\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} \ldots a_{in} & b_1 \\ 0 & b_{22} & b_{23} \ldots b_{2n} & c_2 \\ 0 & 0 & c_{33} \ldots c_{3n} & d_3 \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ 0 & 0 & c_{n3} \ldots c_{nn} & d_n \end{array}\right) \qquad \text{...(5)}$$

Now taking c_{33} as the pivot, using elementary operations, we make all elements below c_{33} as zeros. Continuing the process, all elements below the leading diagonal elements of A are made to zero.

Hence, we get (A, B) after all these operations as

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & \cdots & & a_{1n} & b_1 \\ 0 & b_{22} & b_{23} & \cdots & & b_{2n} & c_2 \\ 0 & 0 & c_{33} & c_{34} \cdots & & c_{3n} & d_3 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & & \alpha_{nn} & K_n \end{array}\right) \qquad \text{...(6)}$$

From, (6), the given system of linear equations is equivalent to

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = [illegible]$$

$$b_{22}x_2 + b_{23}x_3 + \cdots + b_{2n}x_n = [illegible]$$

$$c_{33}x_3 + \cdots c_{3n}\, x_n = d_3$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\alpha_{nn}x_n = K_n$$

Going from the bottom of these equation, we solve for $x_n = \dfrac{K_n}{\alpha_{nn}}$

Using this in the penultimate equation, we get x_{n-1} and so. By this back substitution method, we solve for

$x_n, x_{n-1}, x_{n-2}, \cdots x_2, x_1.$

Note. This method of making the matrix A as upper triangular matrix had been taught in lower classes while finding the rank of the matrix A.

4·2·1 Gauss-Jordan elimination method (Direct method)

This method is a modification of the above Gauss elimination method. In this method, the coefficient matrix A of the system $AX = B$ is brought to a diagonal matrix or unit matrix by making the matrix A not only upper triangular but also lower triangular by making all elements above the leading diagonal of A also as zeros. By this way, the system $AX = B$ will reduce to the form.

$$\left(\begin{array}{ccccc|c} a_{11} & 0 & 0 & 0 & 0 & b_1 \\ 0 & b_{22} & 0 & 0 & 0 & c_2 \\ \cdot & \cdot & \cdot & \cdot & \cdot & d_3 \\ 0 & 0 & 0 & 0 & \alpha_{nn} & K_n \end{array}\right) \qquad \ldots(7)$$

From (7)

$$x_n = \frac{K_n}{\alpha_{nn}}, \ldots, x_2 = \frac{c_2}{b_{22}}, x_1 = \frac{b_1}{a_{11}}$$

Note. By this method, the values of $x_1, x_2, \ldots, x_n$ are got immediately without using the process of back substitution.

Example 1. *Solve the system of equations by (i) Gauss elimination method (ii) Gauss-Jordan method.*

$x + 2y + z = 3, \quad 2x + 3y + 3z = 10; \quad 3x - y + 2z = 13.$ [*MKU 1981*]

Solution. (By Gauss method)

The given system is equivalent to

$$\underset{A}{\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{pmatrix}} \underset{X}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \underset{B}{\begin{pmatrix} 3 \\ 10 \\ 13 \end{pmatrix}}$$

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right) \qquad \ldots(1)$$

Now, we will make the matrix A upper triangular.

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right) \quad \begin{array}{lll} R_2 + (-2) R_1 & i.e., & R_{21}(-2) \\ R_3 + (-3) R_1 & i.e., & R_{31}(-3) \end{array}$$

Now take $b_{22} = -1$ as the pivot and make b_{32} as zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right) \quad R_{32}(-7) \qquad \ldots(2)$$

From this, we get

$$x + 2y + z = 3$$
$$-y + z = 4$$
$$-8z = -24$$

$\therefore \quad z = 3, y = -1, x = 2$ by back substitution.

i.e., $\quad x = 2, y = -1, z = 3$

Solution. (Gauss-Jordan method)

In stage 2, make the element, in the position (1, 2) , also zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -8 & -24 \end{array}\right) \quad R_{12}(2)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & 11 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right) \quad R_3\left(\frac{1}{8}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -3 \end{array}\right) \quad R_{13}(3), R_{23}(1)$$

i.e., $\quad x = 2, -y = 1, -z = -3$

i.e., $\quad x = 2, y = -1, z = 3$

Example 2. *Solve the system by Gauss-Elimination method*

$2x + 3y - z = 5;\quad 4x + 4y - 3z = 3$ *and* $2x - 3y + 2z = 2.$ [*MKU 1980*]

Solution. The system is equivalent to

$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ 2 & -3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}$$

$$A \qquad X = B$$

$$\therefore \qquad (A, B) = \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ 2 & -3 & 2 & 2 \end{array}\right)$$

Step 1. Taking $a_{11} = 2$ as the pivot, reduce all elements below that to zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & -6 & 3 & -3 \end{array}\right) \quad R_{21}(-2), R_{31}(-1)$$

Step 2. Taking the element -2 in the position (2, 2) as pivot, reduce all elements below that to zero.

$$(A, B) \sim \left(\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 0 & -2 & -1 & -7 \\ 0 & 0 & 6 & 18 \end{array}\right) \quad R_{32}(-3)$$

Hence
$$2x + 3y - z = 5$$
$$-2y - z = -7$$
$$6z = 18$$

$\therefore \quad z = 3, y = 2, x = 1.$ by back substitution.

Example 3. *Solve the following system by Gauss-Jordan method:*

$$5x_1 + x_2 + x_3 + x_4 = 4; \quad x_1 + 7x_2 + x_3 + x_4 = 12$$

$$x_1 + x_2 + 6x_3 + x_4 = -5; \quad x_1 + x_2 + x_3 + 4x_4 = -6$$

Solution. Interchange the first and the last equation, so that the coefficient of x_1 in the first equation is 1. Then we have

$$(A, B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & \boxed{6} & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right) \quad R_{21}(-1), R_{31}(-1), R_{41}(-5)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 4 & -6 \\ 0 & \boxed{1} & 0 & -0.5 & 3 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{array}\right) \quad R_2\left(\frac{1}{6}\right) \text{to make the pivot as 1}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & \boxed{5} & -3 & 1 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right) \quad R_{12}(-1), R_{42}(4)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 4.5 & -9 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & \boxed{1} & -0.6 & 0.2 \\ 0 & 0 & -4 & -21 & 46 \end{array}\right) \quad R_3\left(\frac{1}{5}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -23.4 & 46.8 \end{array}\right) \quad R_{13}(-1), R_{43}(4)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 5.1 & -9.2 \\ 0 & 1 & 0 & -0.5 & 3 \\ 0 & 0 & 1 & -0.6 & 0.2 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right) \quad R_4\left(\frac{1}{23.4}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{array}\right) \quad R_{34}\left(-\frac{3}{5}\right), \; R_{24}\left(-\frac{1}{2}\right), R_{14}(5.1)$$

$$x_1 = 1, \; x_2 = 2, \; x_3 = -1, \; x_4 = -2.$$

Example 4. *Solve the system of equations by Gauss-Jordan method :*

$$x + y + z + w = 2$$
$$2x - y + 2z - w = -5$$
$$3x + 2y + 3z + 4w = 7$$
$$x - 2y - 3z + 2w = 5$$

Solution. $(A, B) = \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 2 & -1 & 2 & -1 & -5 \\ 3 & 2 & 3 & 4 & 7 \\ 1 & -2 & -3 & 2 & 5 \end{array}\right)$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & -3 & 0 & -3 & -9 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right) \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - R_1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & \boxed{1} & 0 & 1 & 3 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & -3 & -4 & 1 & 3 \end{array}\right) \quad R_2\left(-\frac{1}{3}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & 4 & 12 \end{array}\right) \quad \begin{array}{l} R_1 + (-1)\, R_2 \\ R_3 + R_2 \\ R_4 + 3R_2 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & -3 \end{array}\right) \quad \begin{array}{l} R_3\left(\frac{1}{2}\right) \\ R_4\left(-\frac{1}{4}\right) \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} \text{Interchanging} \\ R_3 \text{ and } R_4 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 3 \\ 0 & 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad R_1 + (-1)\, R_3$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} R_1 + (-1)\, R_4 \\ R_2 + (-1)\, R_4 \\ R_3 + R_4 \end{array}$$

$\therefore \quad x = 0,\ y = 1,\ z = -1,\ w = 2.$

Example 5. *Apply Gauss-Jordan method to find the solution of the following system :*

$10x + y + z = 12;\ 2x + 10y + z = 13;\ x + y + 5z = 7.$ [*MS 1991*]

Solution. Since the coefficient of x in the last equation is unity, we rewrite the equations interchanging the first and the last. Hence the

$$(A, B) = \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{array}\right) \quad \begin{array}{l} R_2 + (-2) R_1 \\ R_3 + (-10) R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & -9 & -49 & -58 \end{array}\right) \quad R_2\left(\frac{1}{8}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & -\frac{473}{8} & -\frac{473}{8} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 1 & 5 & 7 \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_3\left(-\frac{8}{473}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & \frac{49}{8} & \frac{57}{8} \\ 0 & 1 & -\frac{9}{8} & -\frac{1}{8} \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_1 + (-1)R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad \begin{array}{l} R_2 + \left(\frac{9}{8}\right) R_3 \\ R_1 + \left(-\frac{49}{8}\right) R_3 \end{array}$$

$\therefore \quad x = 1, y = 1, z = 1.$

Example 6. *Using Gauss-Elimination method, solve the system:*

$$3{\cdot}15x - 1{\cdot}96y + 3{\cdot}85z = 12{\cdot}95$$
$$2{\cdot}13x + 5{\cdot}12y - 2{\cdot}89z = -8{\cdot}61$$
$$5{\cdot}92x + 3{\cdot}05y + 2{\cdot}15z = 6{\cdot}88$$

[*MKU 1981*]

Solution. $(A, B) = \left(\begin{array}{ccc|c} 3{\cdot}15 & -1{\cdot}96 & 3{\cdot}85 & 12{\cdot}95 \\ 2{\cdot}13 & 5{\cdot}12 & -2{\cdot}89 & -8{\cdot}61 \\ 5{\cdot}92 & 3{\cdot}05 & 2{\cdot}15 & 6{\cdot}88 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} \boxed{3.15} & -1.96 & 3.85 & 12.95 \\ 0 & \boxed{6.4453} & -5.4933 & -17.3666 \\ 0 & 6.7335 & -5.0855 & -17.4578 \end{array}\right) \begin{array}{l} R_2 + \left(-\frac{2.13}{3.15}\right) R_1 \\ R_3 + \left(\frac{-5.92}{3.15}\right) R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & \boxed{6.4453} & -5.4933 & -17.3666 \\ 0 & 0 & 0.6534 & 0.6853 \end{array}\right)$$

$$\therefore \quad 3.15x - 1.96y + 3.85z = 12.95$$

$$6.4453y - 5.4933z = -17.3666$$

$$0.6534z = 0.6853$$

$$\therefore \quad z = \frac{0.6853}{0.6534} = 1.0488$$

$$y = \frac{5.4933 \times 1.0488 - 17.3666}{6.4453} = -1.8005$$

$$x = \frac{1.96 \times (-1.8005) - 3.85\,(1.0488) + 12.95}{3.15} = 1.7089$$

$\therefore \quad x - 1.7089, y = -1.8005, z = 1.0488.$

Example 7. *Solve by Gauss-Elemination method:*

$3x + 4 \quad + 5z = 18, \quad 2x - y + 8z = 13, \quad 5x - 2y + 7z = 20.$

Solution. $(A, B) = \left(\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{array}\right)$

$$\sim \left(\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -\frac{11}{3} & \frac{14}{3} & 1 \\ 0 & -\frac{26}{3} & -\frac{4}{3} & -10 \end{array}\right) \quad \begin{array}{l} R_2 - \frac{2}{3} R_1 \\ R_3 - \frac{5}{3} R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & 13 & 2 & 15 \end{array}\right) \quad \begin{array}{l} R_2\,(3) \\ R_3\left(-\frac{3}{2}\right) \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & 0 & \frac{204}{11} & \frac{204}{11} \end{array}\right) \quad R_3 + \frac{13}{11} R_2$$

$$\sim \left(\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 0 & -11 & 14 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right) \quad R_3\left(\frac{11}{204}\right)$$

$\therefore \quad z = 1,\ -11y + 14z = 3,\ 3x + 4y + 5z = 18$

Hence, $\quad z = 1,\ y = \dfrac{3 - 14z}{-11} = 1,\ x = \dfrac{18 - 4y - 5z}{3} = 3$

$\therefore \quad x = 3,\ y = 1,\ z = 1.$

EXERCISE 4.1

Solve the following systems by (*i*) Gauss-Elimination (*ii*) Gauss-Jordan methods:

1. $2x + y = 3,\ 7x - 3y = 4$

2. $11x + 3y = 17,\ 2x + 7y = 16$

3. $4x - 3y = 11,\ 3x + 2y = 4$

4. $x - y + z = 1,\ -3x + 2y - 3z = -6,\ 2x - 5y + 4z = 5$

5. $x + 3y + 10z = 24,\ 2x + 17y + 4z = 35,\ 28x + 4y - z = 32$ [*MS Ap 1992*]

6. $x - 3y - z = -30,\ 2x - y - 3z = 5,\ 5x - y - 2z = 142$

7. $5x - 9y - 2z + 4w = 7,\ 3x + y + 4z + 11w = 2,$
$10x - 7y + 3z + 5w = 6,\ -6x + 8y - z - 4w = 5$

8. $10x + y + z = 12,\ x + 10y + z = 12,\ x + y + 10z = 12$

9. $10x + y + z = 18{\cdot}141,\ x + 10y + z = 28{\cdot}140,\ x + y + 10z = 38{\cdot}139$ [*MS 1991*]

10. $3x + y - z = 3,\ 2x - 8y + z = -5,\ x - 2y + 9z = 8$

11. $3x - y + 2z = 12,\ x + 2y + 3z = 11,\ 2x - 2y - z = 2$

12. $2x - 3y + z = -1,\ x + 4y + 5z = 25,\ 3x - 4y + z = 2$

13. $x + 2y + 3z = 6,\ 2x + 4y + z = 7,\ 3x + 2y + 9z = 14$

14. $2x - y + 3z + w = 9,\ 3x + y - 4z + 3w = 3,$
$5x - 4y + 3z - 6w = 2,\ x - 2y - z + 2w = -2$

15. $4x + y + 3z = 11,\ 3x + 4y + 2z = 11,\ 2x + 3y + z = 7$

16. $x + y + 2z = 4,\ 3x + y - 3z = -4,\ 2x - 3y - 5z = -5$

17. $2x + 6y - z = -12,\ 5x - y + z = 11,\ 4x - y + 3z = 10$ [*MS Ap 87*]

18. $x + 2y + z - w = -2,\ 2x + 3y - z + 2w = 7$
$x + y + 3z - 2w = -6,\ x + y + z + w = 2$ [*MS Nov 86*]

19. $4{\cdot}12x - 9{\cdot}68y + 2{\cdot}01z = 4{\cdot}93$
$1{\cdot}88x - 4{\cdot}62y + 5{\cdot}50z = 3{\cdot}11$
$1{\cdot}10x - 0{\cdot}96y + 2{\cdot}72z = 4{\cdot}02$

20. $6x - y + z = 13,\ x + y + z = 9,\ 10x + y - z = 19$

21. $x + 2y - 12z + 8w = 27,\ 5x + 4y + 7z - 2w = 4,$
$6x - 12y - 8z + 3w = 49,\ 3x - 7y - 9z - 5w = -11$

22. $x + 0{\cdot}5y + 0{\cdot}33z = 1,\ 0{\cdot}33x + 0{\cdot}25y + 0{\cdot}2z = 0,\ 0{\cdot}5x + 0{\cdot}33y + 0{\cdot}25z = 0$

23. $2x + 4y + z = 3,\ 3x + 2y - 2z = -2,\ x - y + z = 6$

24. $x + y + z - w = 2,\ 7x + y + 3z + w = 12,$
$8x - y + z - 3w = 5,\ 10x + 5y + 3z + 2w = 20$

25. $2x + 4y + 8z = 41,\ 4x + 6y + 10z = 56,\ 6x + 8y + 10z = 64$

26. $2x + 2y - z + w = 4,\ 4x + 3y - z + 2w = 6,$
$8x + 5y - 3z + 4w = 12,\ 3x + 3y - 2z + 2w = 6$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

will be solvable by this method if

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

In other words, the solution will exist (iteration will converge) if the absolute values of the leading diagonal elements of the coefficient matrix A of the system $AX = B$ are greater than the sum of absolute values of the other coefficients of that row. The condition is *sufficient* but not *necessary*.

4·8 Jacobi method of iteration or Gauss-Jacobi method

Let us explain this method in the case of three equations in three unknowns.

Consider the system of equations,

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2 \qquad \text{...(1)}$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Let us assume $|a_1| > |b_1| + |c_1|$

$$|b_2| > |a_2| + |c_2|$$
$$|c_3| > |a_3| + |b_3|$$

Then, iterative method can be used for the system (1). Solve for x, y, z (whose coefficients are the larger values) in terms of the other variables. That is,

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z)$$
$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \qquad \text{...(2)}$$
$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y)$$

If $x^{(0)}, y^{(0)}, z^{(0)}$ are the initial values of x, y, z respectively, then

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1 y^{(0)} - c_1 z^{(0)})$$
$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(0)} - c_2 z^{(0)}) \qquad \text{...(3)}$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2 x^{(1)} - c_2 z^{(0)})$$

Now, having known $x^{(1)}$ and $y^{(1)}$, use $x^{(1)}$ for x and $y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3 x^{(1)} - b_3 y^{(1)})$$

In finding the values of the unknowns, we use the latest available values on the right hand side. If $x^{(r)}, y^{(r)}, z^{(r)}$ are the rth iterates, then the iteration scheme will be

$$x^{(r+1)} = \frac{1}{a_1}(d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2}(d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3}(d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

This process of iteration is continued until the convergence is assured. As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns, the convergence in Gauss-Seidel method is very fast when compared to Gauss-Jacobi method. The rate of convergence in Gauss-Seidel method is roughly two times than that of Gauss-Jacobi method. As we saw the sufficient conditions already, the sufficient condition for the convergence of this method is also the same as we stated earlier. That is, *the method of iteration will converge if in each equation of the given system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients.* (The largest coefficients must be the coefficients for different unknowns).

Note 1. For all systems of equations, this method will not work (since convergence is not assured). It converges only for special systems of equations.

2. Iteration method is self-correcting method. That is, any error made in computation, is corrected in the subsequent iterations.

3. The iteration is stopped when the values of x, y, z start repeating with the required degree of accuracy.

Example 1. *Solve the following system by Gauss-Jacobi and Gauss-Seidel methods :*

$$10x - 5y - 2z = 3\ ;\quad 4x - 10y + 3z = -3\ ;\quad x + 6y + 10z = -3.$$

[*MS. Ap. 1992*]

Solution. Here, we see that the diagonal elements are dominant. Hence, the iteration process can be applied.

That is, the coefficient matrix $\begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$ is diagonally dominant, since

$|10| > |-5| + |-2|$, $|-10| > |4| + |3|$ and $|10| > |1| + |6|$

Gauss-Jacobi method. Solving for x, y, z, we have

$$x = \frac{1}{10}(3 + 5y + 2z) \qquad \text{...(1)}$$

$$y = \frac{1}{10}(3 + 4x + 3z) \qquad \text{...(2)}$$

$$z = \frac{1}{10}(-3 - x - 6y) \qquad \text{...(3)}$$

First iteration : Let the initial values be (0, 0, 0).

Using these initial values in (1), (2), (3), we get

$$x^{(1)} = \frac{1}{10}[3 + 5(0) + 2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3 + 4(0) + 3(0)] = 0.3$$

$$z^{(1)} = \frac{1}{10}[-3 - (0) - 6(0)] - 0.3$$

Second iteration : Using these values in (1), (2), (3), we get

$$x^{(2)} = \frac{1}{10}[3 + 5\,(0.3) + 2\,(-0.3)] = 0.39$$

$$y^{(2)} = \frac{1}{10}[3 + 4\,(0.3) + 3\,(-0.3)] = 0.33$$

$$z^{(2)} = \frac{1}{10}[-3 - (0.3) - 6\,(0.3)] = -0.51$$

Third iteration: Using the values of $x^{(2)}, y^{(2)}, z^{(2)}$ in (1), (2), (3) we, get

$$x^{(3)} = \frac{1}{10}[3 + 5\,(0.33) + 2\,(-0.51)] = 0.363$$

$$y^{(3)} = \frac{1}{10}[3 + 4\,(0.39) + 3\,(-0.51)] = 0.303$$

$$z^{(3)} = \frac{1}{10}[-3 - (0.39) - 6\,(0.33)] = -0.537$$

Fourth iteration :

$$x^{(4)} = \frac{1}{10}[3 + 5\,(0.303) + 2\,(-0.537)] = 0.3441$$

$$y^{(4)} = \frac{1}{10}[3 + 4\,(0.363) + 3\,(-0.537)] = 0.2841$$

$$z^{(4)} = \frac{1}{10}[-3 - 0{\cdot}363 - 6\,(0{\cdot}303)] = -0{\cdot}5181$$

Fifth interation :

$$x^{(5)} = \frac{1}{10}[3 + 5\,(0{\cdot}2841) + 2\,(-0{\cdot}5181)] = 0{\cdot}33843$$

$$y^{(5)} = \frac{1}{10}[3 + 4\,(0{\cdot}3441) + 3\,(-0{\cdot}5181)] = 0{\cdot}2822$$

$$z^{(5)} = \frac{1}{10}[-3 - (0{\cdot}3441) - 6\,(0{\cdot}2841)] = -0{\cdot}50487$$

Sixth interation :

$$x^{(6)} = \frac{1}{10}[3 + 5\,(0{\cdot}2822) + 2\,(-0{\cdot}50487)] = 0{\cdot}340126$$

$$y^{(6)} = \frac{1}{10}[3 + 4\,(0{\cdot}33843) + 3\,(-0{\cdot}50487)] = 0{\cdot}283911$$

$$z^{(6)} = \frac{1}{10}[-3 - (0{\cdot}33843) - 6\,(0{\cdot}2822)] = -0{\cdot}503163$$

Seventh iteration :

$$x^{(7)} = \frac{1}{10}[3 + 5\,(0{\cdot}283911) + 2\,(-0{\cdot}503163)] = 0{\cdot}3413229$$

$$y^{(7)} = \frac{1}{10}[3 + 4\,(0{\cdot}340126) + 3\,(-0{\cdot}503163)] = 0{\cdot}2851015$$

$$z^{(7)} = \frac{1}{10}[-3 - (0{\cdot}340126) - 6\,(0{\cdot}283911)] = -0{\cdot}5043592$$

Eighth iteration :

$$x^{(8)} = \frac{1}{10}[3 + 5\,(0{\cdot}2851015) + 2\,(-0{\cdot}5043592)] = 0{\cdot}34167891$$

$$y^{(8)} = \frac{1}{10}[3 + 4\,(0{\cdot}3413229) + 3\,(-0{\cdot}5043592)] = 0{\cdot}2852214$$

$$z^{(8)} = \frac{1}{10}[-3 - (0{\cdot}3413229) - 6\,(0{\cdot}2851015)] = -0{\cdot}50519319$$

Nineth iteration :

$$x^{(9)} = \frac{1}{10}[3 + 5\,(0{\cdot}2852214) + 2\,(-0{\cdot}50519319)] = 0{\cdot}341572062$$

$$y^{(9)} = \frac{1}{10}[3 + 4\,(0{\cdot}34167891) + 3\,(-0{\cdot}50519319)] = 0{\cdot}285113607$$

$$z^{(9)} = \frac{1}{10}[-3 - (0{\cdot}34167891) - 6\,(0{\cdot}2852214)] = -0{\cdot}505300731$$

Hence correct to 3 decimal places, the values are

$x = 0{\cdot}342,\ y = 0{\cdot}285,\ z = -0{\cdot}505$

Gauss-Seidel method : Initial values : $y = 0,\ z = 0.$

First iteration :

$$x^{(1)} = \frac{1}{10}\,[3 + 5(0) + 2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}\,[3 + 4\,(0.3) + 3(0)] = 0.42$$

$$z^{(1)} = \frac{1}{10}\,[-3 - (0.3) - 6\,(0.42)] = -0.582$$

Second iteration :

$$x^{(2)} = \frac{1}{10}\,[3 + 5\,(0.42) + 2\,(-0.582)] = 0.3936$$

$$y^{(2)} = \frac{1}{10}\,[3 + 4\,(0.3936) + 3\,(-0.582)] = 0.28284$$

$$z^{(2)} = \frac{1}{10}\,[-3 - (0.3936) - 6\,(0.28284)] = -0.509064$$

Third iteration :

$$x^{(3)} = \frac{1}{10}\,[3 + 5\,(0.28284) + 2\,(-0.509064)] = 0.3396072$$

$$y^{(3)} = \frac{1}{10}\,[3 + 4\,(0.3396072) + 3\,(-0.509064)] = 0.28312368$$

$$z^{(3)} = \frac{1}{10}\,[-3 - (0.3396072) - 6\,(0.28312368)] = -0.503834928$$

Fourth iteration :

$$x^{(4)} = \frac{1}{10}\,[3 + 5\,(0.28312368) + 2\,(-0.503834928)] = 0.34079485$$

$$y^{(4)} = \frac{1}{10}\,[3 + 4\,(0.34079485) + 3\,(-0.50383492)] = 0.285167464$$

$$z^{(4)} = \frac{1}{10}\,[-3 - (0.34079485) - 6\,(0.28516746)] = -0.50517996$$

Fifth iteration :

$$x^{(5)} = \frac{1}{10}\,[3 + 5\,(0.28516746) + 2\,(-0.50517996)] = 0.34155477$$

$$y^{(5)} = \frac{1}{10}\,[3 + 4\,(0.34155477) + 3\,(-0.50517996)] = 0.28506792$$

$$z^{(5)} = \frac{1}{10}\,[-3 - (0.34155477) - 6\,(0.28506792)] = -0.505196229$$

Sixth iteration :

$$x^{(6)} = \frac{1}{10}\,[3 + 5\,(0.28506792) + 2\,(-0.505196229)] = 0.341494714$$

$$y^{(6)} = \frac{1}{10}\,[3 + 4\,(0{\cdot}341494714) + 3\,(-\,0{\cdot}505196229)] = 0{\cdot}285039017$$

$$z^{(6)} = \frac{1}{10}\,[-\,3 - (0{\cdot}341494714) - 6\,(0{\cdot}285039017)] = -\,0{\cdot}5051728$$

Seventh iteration :

$$x^{(7)} = \frac{1}{10}\,[3 + 5\,(0{\cdot}285039017) + 2\,(-\,0{\cdot}5051728)] = 0{\cdot}3414849$$

$$y^{(7)} = \frac{1}{10}\,[3 + 4\,(0{\cdot}3414849) + 3\,(-\,0{\cdot}5051728)] = 0{\cdot}28504212$$

$$z^{(7)} = \frac{1}{10}\,[-\,3 - (0{\cdot}3414849) - 6\,(0{\cdot}28504212)] = -\,0{\cdot}5051737$$

The values at each iteration by both methods are tabulated below:

Iteration	*Gauss-Jacobi method*			*Gauss-Seidel method*		
	x	y	z	x	y	z
1	0·3	0·3	− 0·3	0·3	0·42	− 0·582
2	0·39	0·33	− 0·51	0·3936	0·28284	− 0·509064
3	0·363	0·303	− 0·537	0·3396072	0·28312364	− 0·503834928
4	0·3441	0·2841	− 0·5181	0·34079485	0·28516746	− 0·50517996
5	0·33843	0·2822	− 0·50487	0·3415547	0·28506792	− 0·505196229
6	0·340126	0·283911	− 0·503163	0·3414947	0·2850390	− 0·5051728
7	0·3413229	0·2851015	− 0·5043592	0·3414849	0·28504212	− 0·5051737
8	0·34167[illegible]	0·2852214	− 0·50519319			
9	0·341572062	0·285113607	− 0·505300731			

The values correct to 3 decimal places are

$$x = 0{\cdot}342,\ y = 0{\cdot}285,\ z = -\,0{\cdot}505$$

Note. After getting the values of the unknowns, substitute these values in the given equations, and check the correctness of the results.

Example 2. *Solve the following system of equations by using Gauss-Jacobi and Gauss-Seidel methods (correct to 3 decimal places) :*

$$8x - 3y + 2z = 20$$
$$4x + 11y - z = 33$$
$$6x + 3y + 12z = 35. \qquad [BR.\ Ap.\ '94]$$

Solution. Since the diagonal elements are dominant in the coefficient matrix, we write x, y, z as follows

$$x = \frac{1}{8}\,[20 + 3y - 2z] \qquad ...(1)$$

$$y = \frac{1}{11}\,[33 - 4x + z] \qquad ...(2)$$

$$z = \frac{1}{12}\,[35 - 6x - 3y] \qquad ...(3)$$

Gauss-Jacobi method :

First iteration: Let the initial values be $x = 0,\ y = 0,\ z = 0$

Using the values $x = 0,\ y = 0,\ z = 0$ in (1), (2), (3) we get,

$$x^{(1)} = \frac{1}{8}[20 + 3(0) - 2(0)] = 2{\cdot}5$$

$$y^{(1)} = \frac{1}{11}[33 - 4(0) + 0] = 3{\cdot}0$$

$$z^{(1)} = \frac{1}{12}[35 - 6(0) - 3(0)] = 2{\cdot}916666$$

Second iteration : Using these values $x^{(1)}, y^{(1)}, z^{(1)}$ again in (1), (2), (3), we get

$$x^{(2)} = \frac{1}{8}[20 + 3\,(3{\cdot}0) - 2\,(2{\cdot}916666)] = 2{\cdot}895833$$

$$y^{(2)} = \frac{1}{11}[33 - 4\,(2{\cdot}5) + (2{\cdot}916666)] = 2{\cdot}356060$$

$$z^{(2)} = \frac{1}{12}[35 - 6\,(2{\cdot}5) - 3\,(3{\cdot}0)] = 0{\cdot}916666$$

Third interation :

$$x^{(3)} = \frac{1}{8}[20 + 3\,(2{\cdot}356060) - 2\,(0{\cdot}916666)] = 3{\cdot}154356$$

$$y^{(3)} = \frac{1}{11}[33 - 4\,(2{\cdot}895833) + (0{\cdot}916666)] = 2{\cdot}030303$$

$$z^{(3)} = \frac{1}{12}[35 - 6\,(2{\cdot}895833) - 3\,(2{\cdot}356060)] = 0{\cdot}879735$$

Fourth iteration :

$$x^{(4)} = \frac{1}{8}[20 + 3\,(2{\cdot}030303) - 2\,(0{\cdot}879735)] = 3{\cdot}041430$$

$$y^{(4)} = \frac{1}{11}[33 - 4\,(3{\cdot}154356) + (0{\cdot}879735)] = 1{\cdot}932937$$

$$z^{(4)} = \frac{1}{12}[35 - 6\,(3{\cdot}154356) - 3\,(2{\cdot}030303)] = 0{\cdot}831913$$

Fifth iteration :

$$x^{(5)} = \frac{1}{8}[20 + 3\,(1{\cdot}932937) - 2\,(0{\cdot}831913)] = 3{\cdot}016873$$

$$y^{(5)} = \frac{1}{11}[33 - 4\,(3{\cdot}041430) + (0{\cdot}831913)] = 1{\cdot}969654$$

$$z^{(5)} = \frac{1}{12}[35 - 6\,(3{\cdot}041430) - 3\,(1{\cdot}932937)] = 0{\cdot}912717$$

Sixth iteration :

$$x^{(6)} = \frac{1}{8}[20 + 3\,(1{\cdot}969654) - 2\,(0{\cdot}912717)] = 3{\cdot}010441$$

$$y^{(6)} = \frac{1}{11}[33 - 4(3{\cdot}016873) + (0{\cdot}912717)] = 1{\cdot}985930$$

$$z^{(6)} = \frac{1}{12}[35 - 6(3{\cdot}016873) - 3(1{\cdot}969654)] = 0{\cdot}915817$$

Seventh iteration :

$$x^{(7)} = \frac{1}{8}[20 + 3(1{\cdot}985930) - 2(0{\cdot}915817)] = 3{\cdot}015770$$

$$y^{(7)} = \frac{1}{11}[33 - 4(3{\cdot}010441) + (0{\cdot}915817)] = 1{\cdot}988550$$

$$z^{(7)} = \frac{1}{12}[35 - 6(3{\cdot}010441) - 3(1{\cdot}985930)] = 0{\cdot}914964$$

Eigth iteration :

$$x^{(8)} = \frac{1}{8}[20 + 3(1{\cdot}988550) - 2(0{\cdot}914964)] = 3{\cdot}016946$$

$$y^{(8)} = \frac{1}{11}[33 - 4(3{\cdot}015770) + (0{\cdot}914964)] = 1{\cdot}986535$$

$$z^{(8)} = \frac{1}{12}[35 - 6(3{\cdot}015770) - 3(1{\cdot}988550)] = 0{\cdot}911644$$

Ninth iteration :

$$x^{(9)} = \frac{1}{8}[20 + 3(1{\cdot}986535) - 2(0{\cdot}911644)] = 3{\cdot}017039$$

$$y^{(9)} = \frac{1}{11}[33 - 4(3{\cdot}016946) + (0{\cdot}911644)] = 1{\cdot}985805$$

$$z^{(9)} = \frac{1}{12}[35 - 6(3{\cdot}016946) - 3(1{\cdot}986535)] = 0{\cdot}911560$$

Tenth iteration :

$$x^{(10)} = \frac{1}{8}[20 + 3(1{\cdot}985805) - 2(0{\cdot}911560)] = 3{\cdot}016786$$

$$y^{(10)} = \frac{1}{11}[33 - 4(3{\cdot}017039) + (0{\cdot}911560)] = 1{\cdot}985764$$

$$z^{(10)} = \frac{1}{12}[35 - 6(3{\cdot}017039) - 3(1{\cdot}985805)] = 0{\cdot}911696$$

In 8th, 9th and 10th iterations the values of x, y, z are same correct to 3 decimal places. Hence we stop at this level.

Gauss-Seidel method :

We take the initial values as $y = 0, z = 0$ and use equations (1)

First iteration :

$$x^{(1)} = \frac{1}{8}[20 + 3(0) - 2(0)] = 2{\cdot}5$$

$$y^{(1)} = \frac{1}{11}\,[33 - 4\,(2{\cdot}5) + 0] = 2{\cdot}090909$$

$$z^{(1)} = \frac{1}{12}\,[35 - 6\,(2{\cdot}5) - 3\,(2{\cdot}090909)] = 1{\cdot}143939$$

Second iteration :

$$x^{(2)} = \frac{1}{8}\,[20 + 3\,(2{\cdot}090909) - 2\,(1{\cdot}143939)] = 2{\cdot}998106$$

$$y^{(2)} = \frac{1}{11}\,[33 - 4\,(2{\cdot}998106) + (1{\cdot}143939)] = 2{\cdot}013774$$

$$z^{(2)} = \frac{1}{12}\,[35 - 6\,(2{\cdot}998106) - 3\,(2{\cdot}013774)] = 0{\cdot}914170$$

Third iteration :

$$x^{(3)} = \frac{1}{8}\,[20 + 3\,(2{\cdot}013774) - 2\,(0{\cdot}914170)] = 3{\cdot}026623$$

$$y^{(3)} = \frac{1}{11}\,[33 - 4\,(3{\cdot}026623) + (0{\cdot}914170)] = 1{\cdot}982516$$

$$z^{(3)} = \frac{1}{12}\,[35 - 6\,(3{\cdot}026623) - 3\,(1{\cdot}982516)] = 0{\cdot}907726$$

Fourth iteration :

$$x^{(4)} = \frac{1}{8}\,[20 + 3\,(1{\cdot}982516) - 2\,(0{\cdot}907726)] = 3{\cdot}016512$$

$$y^{(4)} = \frac{1}{11}\,[33 - 4\,(3{\cdot}016512) + (0{\cdot}907726)] = 1{\cdot}985607$$

$$z^{(4)} = \frac{1}{12}\,[35 - 6\,(3{\cdot}016512) - 3\,(1{\cdot}985607)] = 0{\cdot}912009$$

Fifth iteration :

$$x^{(5)} = \frac{1}{8}\,[20 + 3\,(1{\cdot}985607) - 2\,(0{\cdot}912009)] = 3{\cdot}016600$$

$$y^{(5)} = \frac{1}{11}\,[33 - 4\,(3{\cdot}016600) + (0{\cdot}912009)] = 1{\cdot}985964$$

$$z^{(5)} = \frac{1}{12}\,[35 - 6\,(3{\cdot}016600) - 3\,(1{\cdot}985964)] = 0{\cdot}911876$$

Sixth iteration :

$$x^{(6)} = \frac{1}{8}\,[20 + 3\,(1{\cdot}985964) - 2\,(0{\cdot}911876)] = 3{\cdot}016767$$

$$y^{(6)} = \frac{1}{11}\,[33 - 4\,(3{\cdot}016767) + (0{\cdot}911876)] = 1{\cdot}985892$$

$$z^{(6)} = \frac{1}{12}\,[35 - 6\,(3{\cdot}016767) - 3\,(1{\cdot}985892)] = 0{\cdot}911810$$

(The values of x, y, z got by Jacobi method correct to 3 decimal

places are got even in the 6th iteration by Gauss-Seidel method.)

Seventh iteration :

$$x^{(7)} = \frac{1}{8}[20 + 3(1{\cdot}985892) - 2(0{\cdot}911810)] = 3{\cdot}016757$$

$$y^{(7)} = \frac{1}{11}[33 - 4(3{\cdot}016757) + (0{\cdot}911810)] = 1{\cdot}985889$$

$$z^{(7)} = \frac{1}{12}[35 - 6(3{\cdot}016757) - 3(1{\cdot}985889)] = 0{\cdot}911816$$

Since the seventh and eighth iterations give the same values for x, y, z correct to 4 decimal places, we stop here.

$\therefore \quad x = 3{\cdot}0168,\ y = 1{\cdot}9859,\ z = 0{\cdot}9118$

The values of x, y, z by both methods at each iteration are tabulated below:

Iteration	*Gauss-Jacobi method*			*Gauss-Seidel method*		
	x	y	z	x	y	z
1	2·5	3·0	2·916666	2·5	2·090909	1·143939
2	2·895833	2·356060	0·916666	2·998106	2·013774	0·914170
3	3·154356	2·030303	0·879735	3·026623	1·982516	0·907726
4	3·041430	1·932937	0·831913	3·016512	1·985607	0·912009
5	3·016873	1·969654	0·912717	3·016600	1·985964	0·911876
6	3·010441	1·985930	0·915817	3·016767	1·985892	0·911810
7	3·015770	1·988550	0·914964	3·016757	1·985889	0·911816
8	3·016946	1·986535	0·911644			
9	3·017039	1·985805	0·911560			
10	3·016786	1·985764	0·911696			

This shows that the convergence is rapid in Gauss-Seidel method when compared to Gauss-Jacobi method. We see that 10 iterations are necessary in Jacobi method to get the same accuracy as got by 7 iterations in Gauss-Seidel method.

Example 3. *Solve the following system of equations by Gauss-Jacobi and Gauss-Seidel method correct to three decimal places :*

$$x + y + 54z = 110$$
$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$

Solution. As the coefficient matrix is not diagonally dominant as it is, we rewrite the equation, as noted below, so that the coefficient matrix becomes diagonally dominant

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

$$y^{(1)} = \frac{1}{11}\,[33 - 4\,(2{\cdot}5) + 0] = 2{\cdot}090909$$

$$z^{(1)} = \frac{1}{12}\,[35 - 6\,(2{\cdot}5) - 3\,(2{\cdot}090909)] = 1{\cdot}143939$$

Second iteration :

$$x^{(2)} = \frac{1}{8}\,[20 + 3\,(2{\cdot}090909) - 2\,(1{\cdot}143939)] = 2{\cdot}998106$$

$$y^{(2)} = \frac{1}{11}\,[33 - 4\,(2{\cdot}998106) + (1{\cdot}143939)] = 2{\cdot}013774$$

$$z^{(2)} = \frac{1}{12}\,[35 - 6\,(2{\cdot}998106) - 3\,(2{\cdot}013774)] = 0{\cdot}914170$$

Third iteration :

$$x^{(3)} = \frac{1}{8}\,[20 + 3\,(2{\cdot}013774) - 2\,(0{\cdot}914170)] = 3{\cdot}026623$$

$$y^{(3)} = \frac{1}{11}\,[33 - 4\,(3{\cdot}026623) + (0{\cdot}914170)] = 1{\cdot}982516$$

$$z^{(3)} = \frac{1}{12}\,[35 - 6\,(3{\cdot}026623) - 3\,(1{\cdot}982516)] = 0{\cdot}907726$$

Fourth iteration :

$$x^{(4)} = \frac{1}{8}\,[20 + 3\,(1{\cdot}982516) - 2\,(0{\cdot}907726)] = 3{\cdot}016512$$

$$y^{(4)} = \frac{1}{11}\,[33 - 4\,(3{\cdot}016512) + (0{\cdot}907726)] = 1{\cdot}985607$$

$$z^{(4)} = \frac{1}{12}\,[35 - 6\,(3{\cdot}016512) - 3\,(1{\cdot}985607)] = 0{\cdot}912009$$

Fifth iteration :

$$x^{(5)} = \frac{1}{8}\,[20 + 3\,(1{\cdot}985607) - 2\,(0{\cdot}912009)] = 3{\cdot}016600$$

$$y^{(5)} = \frac{1}{11}\,[33 - 4\,(3{\cdot}016600) + (0{\cdot}912009)] = 1{\cdot}985964$$

$$z^{(5)} = \frac{1}{12}\,[35 - 6\,(3{\cdot}016600) - 3\,(1{\cdot}985964)] = 0{\cdot}911876$$

Sixth iteration :

$$x^{(6)} = \frac{1}{8}\,[20 + 3\,(1{\cdot}985964) - 2\,(0{\cdot}911876)] = 3{\cdot}016767$$

$$y^{(6)} = \frac{1}{11}\,[33 - 4\,(3{\cdot}016767) + (0{\cdot}911876)] = 1{\cdot}985892$$

$$z^{(6)} = \frac{1}{12}\,[35 - 6\,(3{\cdot}016767) - 3\,(1{\cdot}985892)] = 0{\cdot}911810$$

(The values of x, y, z got by Jacobi method correct to 3 decimal

places are got even in the 6th iteration by Gauss-Seidel method.)

Seventh iteration :

$$x^{(7)} = \frac{1}{8}[20 + 3(1\cdot985892) - 2(0\cdot911810)] = 3\cdot016757$$

$$y^{(7)} = \frac{1}{11}[33 - 4(3\cdot016757) + (0\cdot911810)] = 1\cdot985889$$

$$z^{(7)} = \frac{1}{12}[35 - 6(3\cdot016757) - 3(1\cdot985889)] = 0\cdot911816$$

Since the seventh and eighth iterations give the same values for x, y, z correct to 4 decimal places, we stop here.

$\therefore \quad x = 3\cdot0168,\ y = 1\cdot9859,\ z = 0\cdot9118$

The values of x, y, z by both methods at each iteration are tabulated below:

Iteration	*Gauss-Jacobi method*			*Gauss-Seidel method*		
	x	y	z	x	y	z
1	2·5	3·0	2·916666	2·5	2·090909	1·143939
2	2·895833	2·356060	0·916666	2·998106	2·013774	0·914170
3	3·154356	2·030303	0·879735	3·026623	1·982516	0·907726
4	3·041430	1·932937	0·831913	3·016512	1·985607	0·912009
5	3·016873	1·969654	0·912717	3·016600	1·985964	0·911876
6	3·010441	1·985930	0·915817	3·016767	1·985892	0·911810
7	3·015770	1·988550	0·914964	3·016757	1·985889	0·911816
8	3·016946	1·986535	0·911644			
9	3·017039	1·985805	0·911560			
10	3·016786	1·985764	0·911696			

This shows that the convergence is rapid in Gauss-Seidel method when compared to Gauss-Jacobi method. We see that 10 iterations are necessary in Jacobi method to get the same accuracy as got by 7 iterations in Gauss-Seidel method.

Example 3. *Solve the following system of equations by Gauss-Jacobi and Gauss-Seidel method correct to three decimal places :*

$$x + y + 54z = 110$$
$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$

Solution. As the coefficient matrix is not diagonally dominant as it is, we rewrite the equation, as noted below, so that the coefficient matrix becomes diagonally dominant

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

Solving for x, y, z, we get

$$x = \frac{1}{27}[85 - 6y + z] \qquad \text{...(1)}$$

$$y = \frac{1}{15}[72 - 6x - 2z] \qquad \text{...(2)}$$

$$z = \frac{1}{54}[110 - x - y] \qquad \text{...(3)}$$

Starting with the initial value $x = 0$, $y = 0$, $z = 0$ and using (1), (2), (3) and repeating the process we get the values of x, y, z as the tabulated by both methods. (Gauss-Jacobi and Gauss-Seidel)

Iteration	*Gauss-Jacobi method*			*Gauss-Seidel method*		
	x	y	z	x	y	z
1	3·14815	4·8	2·03704	3·14815	3·54074	1·91317
2	2·15693	3·26913	1·88985	2·43218	3·57204	1·92585
3	2·49167	3·68525	1·93655	2·42569	3·57294	1·92595
4	2·40093	3·54513	1·92265	2·42549	3·57301	1·92595
5	2·43155	3·58327	1·92692	2·42548	3·57301	1·92595
6	2·42323	3·57046	1·92565	2·42548	3·57301	1·92595
7	2·42603	3·57395	1·92604			
8	2·42527	3·57278	1·92593			

Hence $x = 2{\cdot}425$, $y = 3{\cdot}573$, and $z = 1{\cdot}926$

(correct to 3 decimal places)

Example 4. *Solve, by Gauss-Seidel method, the following system:*

$$28x + 4y - z = 32$$
$$x + 3y + 10z = 24$$
$$2x + 17y + 4z = 35$$

Solution. Since the diagonal elements in the coefficient matrix are not dominant, we rearrange the equations, as follows, such that the elements in the coefficient matrix are dominant.

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

Hence,
$$x = \frac{1}{28}[32 - 4y + z]$$

$$y = \frac{1}{17}[35 - 2x - 4z]$$

$$z = \frac{1}{10}[24 - x - 3y]$$

Setting $y = 0$, $z = 0$, we get

First iteration :

$$x^{(1)} = \frac{1}{28}[32 - 4(0) + 0] = 1{\cdot}1429$$

$$y^{(1)} = \frac{1}{17}[35 - 2(1{\cdot}1429) - 4(0)] = 1{\cdot}9244$$

$$z^{(1)} = \frac{1}{10}[24 - 1{\cdot}1429 - 3(1{\cdot}9244)] = 1{\cdot}8084$$

Second iteration :

$$x^{(2)} = \frac{1}{28}[32 - 4(1{\cdot}9244) + 1{\cdot}8084] = 0{\cdot}9325$$

$$y^{(2)} = \frac{1}{17}[35 - 2(0{\cdot}9325) - 4(1{\cdot}8084)] = 1{\cdot}5236$$

$$z^{(3)} = \frac{1}{10}[24 - 0{\cdot}9325 - 3(1{\cdot}5236)] = 1{\cdot}8497$$

Third iteration :

$$x^{(3)} = \frac{1}{28}[32 - 4(1{\cdot}5236) + 1{\cdot}8497] = 0{\cdot}9913$$

$$y^{(3)} = \frac{1}{17}[35 - 2(0{\cdot}9913) - 4(1{\cdot}8497)] = 1{\cdot}5070$$

$$z^{(3)} = \frac{1}{10}[24 - 0{\cdot}9913 - 3(1{\cdot}5070)] = 1{\cdot}8488$$

Fourth iteration :

$$x^{(4)} = \frac{1}{28}[32 - 4(1{\cdot}5070) + 1{\cdot}8488] = 0{\cdot}9936$$

$$y^{(4)} = \frac{1}{17}[35 - 2(0{\cdot}9936) - 4(1{\cdot}8488)] = 1{\cdot}5069$$

$$z^{(4)} = \frac{1}{10}[24 - 0{\cdot}9936 - 3(1{\cdot}5069)] = 1{\cdot}8486$$

Fifth iteration :

$$x^{(5)} = \frac{1}{28}[32 - 4(1{\cdot}5069) + (1{\cdot}8486)] = 0{\cdot}9936$$

$$y^{(6)} = \frac{1}{17}[35 - 2(0{\cdot}9936) - 4(1{\cdot}8486)] = 1{\cdot}5069$$

$$z^{(6)} = \frac{1}{10}[24 - 0{\cdot}9936 - 3(1{\cdot}5069)] = 1{\cdot}8486$$

Since the values of x, y, z in the 4th and 5th iterations are same, we stop the process here.

Hence, $x = 0{\cdot}9936,\ y = 1{\cdot}5069,\ z = 1{\cdot}8486$

EXERCISE 4.6

Solve the following system of equations by (*i*) Gauss-Jacobi method and (*ii*) Gauss-Seidel method :

1. $5x - 2y + z = -4,\ x + 6y - 2z = -1$, and $3x + y + 5z = 13$
2. $8x + y + z = 8,\ 2x + 4y + z = 4,\ x + 3y + 3z = 5$
3. $8x - 6y + z = 13{\cdot}67,\ 3x + y - 2z = 17{\cdot}59,\ 2x - 6y + 9z = 29{\cdot}29$
4. $30x - 2y + 3z = 75,\ 2x + 2y + 18z = 30,\ x + 17y - 2z = 48$

   [**Hint.** Interchange second and third equations.]
5. $y - x + 10z = 35{\cdot}61,\ x + z + 10y = 20{\cdot}08,\ y - z + 10x = 11{\cdot}19$
6. $$\begin{aligned} 3{\cdot}122x + 0{\cdot}5756y - 0{\cdot}1565z - 0{\cdot}0067t &= 1{\cdot}571 \\ 0{\cdot}5756x + 2{\cdot}938y + 0{\cdot}1103z - 0{\cdot}0015t &= -0{\cdot}9275 \\ -0{\cdot}1565x + 0{\cdot}1103y + 4{\cdot}127z + 0{\cdot}2051t &= -0{\cdot}0652 \\ -0{\cdot}0067x - 0{\cdot}0015y + 0{\cdot}2051z + 4{\cdot}133t &= -0{\cdot}0178 \end{aligned}$$
7. $10x - 2y + z = 12,\ x + 9y - z = 10,\ 2x - y + 11z = 20$
8. $10x - 2y - z - t = 3,\ -2x + 10y - z - t = 15$

   $-x - y + 10z - 2t = 27,\ -x - y - 2z + 10t = -9$
9. $8x - y + z = 18,\ 2x + 5y - 2z = 3,\ x + y - 3z = -16$
10. $2x + y + z = 4,\ x + 2y + z = 4,\ x + y + 2z = 4$ [*MS. Nov. '87*]
11. $4x + 2y + z = 14,\ x + 5y - z = 10,\ x + y + 8z = 20$ [*Ms. Nov. 86*]
12. $8x + y + z = 8,\ 2x + 4y + z = 4,\ x + 3y + 5z = 5$ [*Ms. Ap. 92*]
13. $14x - 5y = 5{\cdot}5,\ 2x + 7y = 19{\cdot}3$
14. $x - 2y + 10z = 30{\cdot}6,\ 2x + 5y - z = 10{\cdot}5,\ 3x + y + z = 9{\cdot}3$
15. $8x - 6y + z = 13{\cdot}67,\ 3x + 11y - 2z = 17{\cdot}59,\ 2x - 6y + 9z = 29{\cdot}29$
16. $7{\cdot}6x - 2{\cdot}4y + 1{\cdot}3z = 20{\cdot}396,\ 3{\cdot}7x + 7{\cdot}9y - 2{\cdot}5z = 35{\cdot}866,$

    $1{\cdot}9x - 4{\cdot}3y + 8{\cdot}2z = 32{\cdot}514$
17. $83x + 11y - 4z = 95,\ 7x + 52y + 13z = 104,\ 3x + 8y + 29z = 71$

4·10. Relaxation methods

We will consider a system of three equations in three unknowns as given below for the sake of simplicity. The method is applicable even for more number of equations.

Consider the system of equations,

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \qquad \ldots(1)$$

We define the residuals r_1, r_2, r_3 by the relations

$$\left.\begin{aligned} r_1 &= a_1x + b_1y + c_1z - d_1 \\ r_2 &= a_2x + b_2y + c_2z - d_2 \\ r_3 &= a_3x + b_3y + c_3z - d_3 \end{aligned}\right\} \qquad \ldots(2)$$