AL-Karkh University of Science
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Numerical Methods

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Department of Geophysics
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References:

1- 1- Mathematical Methods Dr. T. K. V. Iyengar , Dr. B Krishna Gandhi (2008)

15

Numerical Techniques

15.1 INTRODUCTION

In this chapter, we shall deal with the methods for solving the equations. Sometimes, a rough approximation of a root can be found by graph and more accurate results by the following methods :

(*i*) Newton Raphson method or successive substitution method.

(*ii*) Rule of false position *(Regula falsi)*.

(*iii*) Iteration method.

15.2 SOLUTION OF THE EQUATIONS GRAPHICALLY

Step 1. Find a small interval (a, b) between which the root of the equation lies.

Let $$f(x) = 0 \quad \text{...(1)}$$

and $$f(a) = -\text{ve} \text{ and } f(b) = +\text{ve}$$

then the root of the equation (1) lies between a and b.

For example
$$f(x) = 2x^2 + x - 15 = 0$$
$$f(2) = 8 + 2 - 15 = -5 = -\text{ve}$$
$$f(3) = 18 + 3 - 15 = +6 = +\text{ve}$$

∴ The root of thc equation lies between 2 and 3.

Step 2. Write the equation $f(x) = 0$ as $\phi(x) = \psi(x)$

For example $\quad 2x^2 + x - 15 = 0 \quad$ or $\quad 2x^2 = 15 - x$

Step 3. Prepare two tables for $y = \phi(x)$ and $y = \psi(x)$ taking values of x between a and b.

Step 4. Plot these points and join them to get smooth curves.

Step 5. Note down the abscissa of the point of intersection of the curves $y = \phi(x)$ and $y = \psi(x)$. This is the required root of the equation $f(x) = 0$.

Note. Sometimes we do not write $f(x) = 0$ as $\phi(x) = \psi(x)$. We adopt the following method :

(*i*) Find a small interval (a, b) between which the root lies. $f(a)$ and $f(b)$ are of opposite sign.

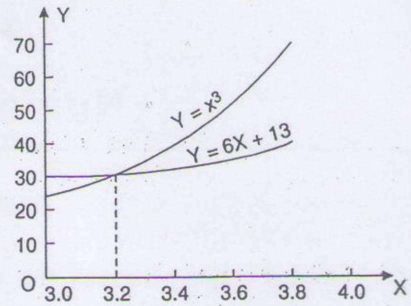
(*ii*) Prepare a table of the different values of x between a and b, for $y = f(x)$.

(*iii*) Plot these points and join them to get smooth curve.

(*iv*) The real root of the equation $f(x) = 0$ is the abscissa where the curve cuts the x-axis Note it.

Example 1. *Find graphically the positive root of the equation.*

$$x^3 - 6x - 13 = 0.$$



Solution.

$f(x) = x^3 - 6x - 13 = 0$ ...(1)

$f(3) = 27 - 18 - 13 = -4 = -ve$

$f(4) = 64 - 24 - 13 = 27 = +ve$

The root of (1) lies between 3 and 4 as $f(3)$ and $f(4)$ are opposite in sign.

(1) is written as $x^3 = 6x + 13$, $y = x^3$

and $y = 6x + 13$

Let us draw two curves for $y = x^3$ and $y = 6x + 13$.

$y = x^3$

x	3	3.2	3.4	3.6	3.8	4.0
y	27	32.8	39.3	46.7	54.9	64

$y = 6x + 13$

x	3	3.2	3.4	3.6	3.8	4
y	31	32.2	33.4	34.6	35.8	37

Let the origin be (3, 0).

The graphs of $y = x^3$ and $y = 6x + 13$ are sketched in the figure. The abscissa of the point of intersection of two curves is 3.2.

$\therefore$ The root of the given equation is 3.2. **Ans.**

Example 2. *Solve graphically the equation* $x - 1 = \sin x$.

Solution. $x - 1 = \sin x$

We take two equations $y = x - 1$ and $y = \sin x$. Let us find out the abscissa of the point of intersection of the line $y = x - 1$ and the curve $y = \sin x$ and give a rough estimate of the root.

For the straight line $y = x - 1$, we have the table :

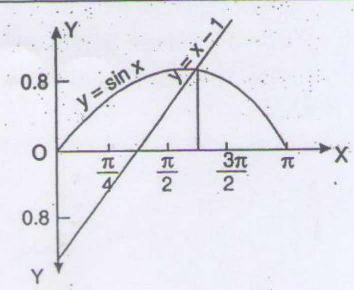
x	0	1	$3\pi/4$
$y = x - 1$	-1	0	1.4

For the sine curve, we have the following table :

x	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
$y = \sin x$	0	0.71	1.00	0.71	0

On the same axes, and with the same scale construct the graphs of $y = x - 1$ and $y = \sin x$.

From the graph, we get $x = 1.95$ radians approximately. **Ans.**



EXERCISE 15.1

1. Draw the graph of $y = x^3$ and $y = -2x + 20$ and find the approximate solution of the equation $x^3 + 2x - 20 = 0$. **Ans.** 2.47
2. Solve graphically $x^3 - 2x - 5 = 0$ **Ans.** 2.099
3. Solve graphically $x^5 - x - 2 = 0$. **Ans.** -0.2
4. Solve graphically $e^{3x} - 5x^2 - 17 = 0$. **Ans.** 1.04
5. Draw the graph of $y = e^{x-1}$ and find graphically the values of the root of the equation $3 - x = e^{x-1}$. **Ans.** 1.44

15.3 NEWTON-RAPHSON METHOD OR SUCCESSIVE SUBSTITUTION METHOD

By this method, we get closer approximation of the root of an equation if we already know its approximate root.

Let the equation be $f(x) = 0$. ...(1)

Let its approximate root be a and better approximate root be $a + h$.

Now we proceed to find h.

$$f(a+h) = 0 \text{ approximately} \quad [\text{as } a + h, \text{ is the root of } f(x) = 0] \quad ...(2)$$

By Taylor's theorem

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a) + \ldots$$

or
$$f(a+h) = f(a) + hf'(a) \quad ...(3)$$

Since h is small, we neglect the h^2 and higher power of h.

From (2) and (3), we have

$$0 = f(a) + hf'(a) \quad \text{or} \quad h = -\frac{f(a)}{f'(a)}$$

or
$$a + h = a - \frac{f(a)}{f'(a)} = a_1 \qquad [\text{First approximate root} = a]$$

Second approximate root $a_2 = a_1 - \dfrac{f(a_1)}{f'(a_1)}$

Similarly third approximate root, $a_3 = a_2 - \dfrac{f(a_2)}{f'(a_2)}$

By repeating this operation, we get closer approximation of the root.

Note. (1) In the beginning, we guess two numbers b and c such that $f(b)$ and $f(c)$ are of opposite sign. Then the first approximate root a lies between b and c.

(2) If $f'(x)$ is zero or nearly zero, this method fails.

Example 3. *Starting with $x_0 = 3$, find a root of $x^3 - 3x - 5 = 0$, correct to three decimal places. Use Newton-Raphson method.*

Solution. $f(x) = x^3 - 3x - 5 = 0, \quad f'(x) = 3x^2 - 3$

$$f(3) = 27 - 9 - 5 = 13, \quad f'(3) = 27 - 3 = 24$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{13}{24} = 3 - 0.5417 = 2.4583$$

$$x_2 = 2.4583 - \frac{f(2.4583)}{f'(2.4583)} = 2.4583 - \frac{2.4812}{15.1297} = 2.4583 - 0.1640 = 2.2943$$

$$x_3 = 2.2943 - \frac{f(2.2943)}{f'(2.2943)} = 2.2943 - \frac{0.1939}{12.7914} = 2.2791$$

$$f(2.2791) = 0.0010$$

Hence the required root $= 2.2791$ **Ans.**

Example 4. *Find the real root of the following equation, correct to three decimal places, using Newton-Raphson method.*

$$x^3 - 2x - 5 = 0$$

Solution. $x^3 - 2x - 5 = 0$ ...(1)

Let
$$f(x) = x^3 - 2x - 5$$
$$f(2) = 8 - 4 - 5 = -1$$
$$f(2.5) = (2.5)^3 - 2(2.5) - 5 = +5.625$$

Since $f(2)$ and $f(2.5)$ are, of opposite sign, the root of (1) lies between 2 and 2.5 ; $f(2)$ is near to zero than $f(2.5)$, so 2 is better appropriate root than 2.5.

$$f'(x) = 3x^2 - 2 \qquad f'(2) = 12 - 2 = 10$$

Let 2 be an approximate root of (1). By Newton-Raphson method

$$a_1 = a - \frac{f(a)}{f'(a)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{-1}{10} = 2.1$$

$$f(2.1) = (2.1)^3 - 2(2.1) - 5 = 9.261 - 4.2 - 5 = 0.061$$

$$f'(2.1) = 3(2.1)^2 - 2 = 11.23$$

$$a_2 = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.1 - \frac{0.061}{11.23} = 2.1 - 0.00543 = 2.09457$$

$$f(2.09457) = (2.09457)^3 - 2(2.09457) - 5$$
$$= 8.80558 - 4.18914 - 5 = -0.38356$$

$$f'(2.09457) = 3(2.09457)^2 - 2 = 13.16167 - 2 = 11.16167$$

$$a_3 = 2.09457 - \frac{f(2.09457)}{f'(2.09457)} = 2.09457 - \frac{-0.38356}{11.16167} = 2.09457 + 0.034364 = 2.128934$$

$$f(2.128934) = 9.649095 - 4.257868 - 5 = +0.391227$$

$$f(2.128934) = 3(2.128934)^2 - 2 = 13.59708 - 2 = 11.59708$$

$$a_4 = 2.128934 - \frac{f(2.128934)}{f'(2.128934)} = 2.128934 - \frac{0.391227}{11.59708} = 2.128934 - 0.03373 = 2.09461$$

Ans.

Example 5. *Find an interval of length 1, in which the root of $f(x) = 3x^3 - 4x^2 - 4x - 7 = 0$ lies. Take the middle point of this interval as the starting approximation and iterate two times, using the Newton- Raphson method.*

Solution. $f(x) = 3x^3 - 4x^2 - 4x - 7 = 0$ ...(1)

$$f(2) = 24 - 16 - 8 - 7 = -7$$
$$f(3) = 81 - 36 - 12 - 7 = +26$$

The root of (1) lies between 2 and 3 as $f(2)$ and $f(3)$ are of opposite sign.

The middle point of this interval is 2.5.

$$f(2.5) = 46.875 - 25 - 10 - 7 = 4.875$$

By Newton-Raphson method

$$a_1 = a - \frac{f(a)}{f'(a)} \quad \text{and} \quad f'(x) = 9x^2 - 8x - 4$$

$$f'(2.5) = 56.25 - 20 - 4 = 32.25$$

$$a_1 = 2.5 - \frac{f(2.5)}{f'(2.5)} = 2.5 - \frac{4.875}{32.25} = 2.5 - 0.15 = 2.35$$

$$f(2.35) = 38.93 - 22.09 - 9.4 - 7 = 0.44$$

$$f'(2.35) = 49.7 - 18.8 - 4 = 26.9$$

$$a_2 = 2.35 - \frac{f(2.35)}{f'(2.35)} = 2.35 - \frac{0.44}{26.9} = 2.35 - 0.016 = 2.334$$

$$f(2.334) = 38.14 - 21.79 - 9.34 - 7 = 0.01$$

Hence the required root is 2.334 **Ans.**

Example 6. *By using Newton-Raphson's method, find the root of $x^4 - x - 10 = 0$, which is near to $x = 2$ correct to three places of decimal.*

Solution. $$f(x) = x^4 - x - 10 = 0, \quad f'(x) = 4x^3 - 1$$

$$f(2) = 16 - 2 - 10 = 4$$

$$f'(2) = 32 - 1 = 31$$

By Newton-Raphson's method

$$a_1 = a - \frac{f(a)}{f'(a)} = 2 - \frac{f(2)}{f'(2)} = 2 - \frac{4}{31} = 2 - .129 = 1.871$$

$$f(1.871) = (1.871)^4 - 1.871 - 10 = 12.25 - 1.871 - 10 = 0.379$$

$$f'(1.871) = 4(1.871)^3 - 1 = 4 \times 6.5497 - 1 = 25.1988$$

$$a_2 = 1.871 - \frac{f(1.871)}{f'(1.871)} = 1.871 - \frac{0.379}{25.1988} = 1.871 - 0.0150 = 1.856$$

$$f(1.856) = (1.856)^4 - (1.856) - 10 = 11.8662 - 11.856 = 0.0102$$

$$f'(1.856) = 4(1.856)^3 - 1 = 4 \times 6.4623 - 1 = 24.8492$$

$$a_3 = 1.856 - \frac{f(1.856)}{f'(1.856)} = 1.856 - \frac{0.0102}{24.8492} = 1.856 - 0.00041 = 1.85559$$

$$f(1.85559) = (1.85559)^4 - 1.85559 - 10 = 11.85572 - 11.85559 = 0.00013$$

$$f'(1.85559) = 4(1.85559)^3 - 1 = 4 \times 6.389193927 - 1 = 24.55677571$$

$$a_4 = 1.85559 - \frac{f(1.85559)}{f'(1.85559)} = 1.85559 - \frac{0.00013}{24.55677571} = 1.85559 - 0.00000529$$

$$= 1.85558471$$ **Ans.**

Example 7. *Determine the root of $x^4 + x^3 - 7x^2 - x + 5 = 0$ which lies between 2 and 3 correct to three decimal places.*

Solution. $$f(x) = x^4 + x^3 - 7x^2 - x + 5 = 0$$

or $$f(2) = 16+8-28-2+5 = -1$$

or $$f(3) = 81+27-63-3+5 = +47.$$

Root lies between 2 and 3.

Taking $x_1 = 2$ as first approximate root.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{f(2)}{f'(2)} \qquad \left[\begin{array}{l} f'(x) = 4x^3+3x^2-14x-1 \\ f'(2) = 32+12-28-1 = 15 \end{array}\right]$$

or $$x_2 = 2 - \frac{-1}{15} = 2\frac{1}{15} = 2.067$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.067 - \frac{f(2.067)}{f'(2.067)}$$

$$= 2.067 - \frac{-0.0028}{18.422} = 2.067 + 0.0001519 = 2.0671519 \qquad \textbf{Ans.}$$

Example 8. *Using Newton-Raphson method evaluate to two decimal figures, the root of the equation $e^x = 3x$ lying between 0 and 1.*

Solution. $$f(x) = e^x - 3x = 0$$

$$f(0) = 1$$

$$f(1) = e^1 - 3 = -0.2817$$

The middle point of the interval (0, 1) is 0.5.

$$f(0.5) = e^{0.5} - 3(0.5) = 1.649 - 1.5 = 0.149$$

$$f'(x) = e^x - 3,\ f'(0.5) = e^{0.5} - 3 = 1.649 - 3 = -1.351$$

By Newton-Raphson method $a_1 = a - \dfrac{f(a)}{f'(a)}$

so $$a_1 = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{0.149}{-1.351} = 0.5 + 0.11 = 0.61$$

$$f(0.61) = e^{0.61} - 3(0.61) = 1.84 - 1.83 = 0.01$$

$$f'(0.61) = e^{0.61} - 3 = 1.84 - 3 = -1.16$$

$$a_2 = 0.61 - \frac{f(0.61)}{f'(0.61)} = 0.61 - \frac{0.01}{-1.16} = 0.61 + 0.0086 = 0.6186$$

$$f(0.6186) = e^{0.6186} - 3(0.6186) = 1.8563 - 1.8558 = 0.0005$$

$$x = 0.6186 \qquad \textbf{Ans.}$$

Example 9. *Compute the real root of $x \log_{10} x - 1.2 = 0$*

Solution. $x \log_{10} x - 1.2 = 0$

Let $$f(x) = x\log_{10} x - 1.2 \text{ or } f(3) = 3\log_{10} 3 - 1.2$$

or $$f(3) = 3 \times 0.4771 - 1.2 = 1.4313 - 1.2 = +0.2313$$

and $$f(2) = 2 \times 0.3010 - 1.2 = 0.6020 - 1.2 = -0.5980$$

$f(3)$ is $+ve$ and $f(2)$ is $-ve$, so the root of the given equation lies between 3 and 2.

$$f(x) = x\log_{10} x - 1.2 = 0.4343\, x \log_e x - 1.2$$

$$f'(x) = 0.4343 \log_e x + 0.4343 = \log_{10} x + 0.4343$$

Taking $x_1 = 3$ as first approximation, we have

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \text{(By Newton's method)}$$

$$x_2 = 3 - \frac{3\log_{10} 3 - 1.2}{\log_{10} 3 + 0.4343} = 3 - \frac{0.2313}{0.9114} = 3 - 0.2538 = 2.7462$$

$$x_3 = 2.7462 - \frac{2.7462\log_{10} 2.7462 - 1.2}{\log_{10} 2.7462 + 0.4343} = 2.7462 - \frac{0.0048}{0.8730} = 2.7407 \qquad \textbf{Ans.}$$

Example 10. *Write the Newton-Raphson procedure for finding $\sqrt[3]{N}$, where N is a real number. Use it to find $\sqrt[3]{18}$ correct to 2 decimals, assuming 2.5 as the initial approximation.*

Solution. Let $x = \sqrt[3]{N} \Rightarrow x^3 = N$ or $x^3 - N = 0$

Let $f(x) = x^3 - N = 0 \Rightarrow f'(x) = 3x^2$

By Newton – Raphson Method, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}, \quad n = 0, 1, 2, \ldots.$$

Let $N = 18$, $x =$ app. cube root of $18 = 2.5$

$$x_1 = \frac{2(2.5)^3 + 18}{3 \times (2.5)^2} = 2.62667$$

Repeat this method.

Exercise 15.2

Solve the following equations by Newton's method:

1. $x^3 - 2x - 5 = 0$ **Ans.** 2.0946
2. $x^3 - 2x + 0.5 = 0$ **Ans.** 0.2578
3. $3x^3 + 8x^2 + 8x + 5 = 0$ **Ans.** -1.67
4. $x^3 - 5x + 3 = 0$ **Ans.** 0.6565
5. $x - 2\sin x = 0$ **Ans.** 1.8955
6. $xe^x - 2 = 0$ **Ans.** 0.853
7. $x^2 - 4\sin x = 0$ **Ans.** 1.9337
8. Apply Newton-Raphson method to find an approximate solution of the equation $e^x - 3^x = 0$ correct upto three significant figures (assume $x = 0.4$ as an approximate root of the equation). **Ans.** 0.619
9. Determine approximately the root of the equation $x + \log_{10} x = 3.375$ correct to two significant figures. **Ans.** 2.911
10. Determine approximately the smallest positive root of the equation $x^2 + 2x - 2 = 0$, correct to two significant figures using Newton-Raphson method. **Ans.** 0.7482
11. Design a Newton-Raphson iteration to compute the cube-root of a positive number, N. Perform two iterations of this method to compute $(2)^{1/3}$ starting from $x_0 = 1$. **Ans.** 1.264
12. A root of the equation $e^x = 1 + x + \dfrac{x^2}{2} + \dfrac{x^2 e^{0.3x}}{6}$ is close to 2.5. Find this root to three decimal places, using Newton-Raphson method. **Ans.** 2.364

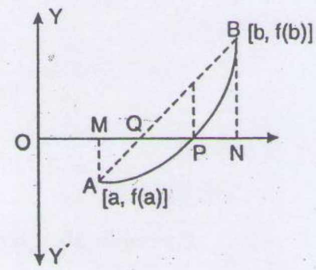
15.4 RULE OF FALSE POSITION (REGULA FALSI)

Let $f(x) = 0$ ...(1)

Let $y = f(x)$ be represented by the curve AB.

The curve AB cuts the x-axis at P.

The real root of (1) is OP.



The false position of the curve AB is taken as the chord AB. The chord AB cuts the x-axis at Q. The approximate root of $f(x) = 0$ is OQ.

By this method, we find OQ.

Let $A\,[a, f(a)]$, $B\,[b, f(b)]$ be the extremities of the chord AB.

The equation of the chord AB is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a) \qquad \text{(Two points form)}$$

To find OQ, put $y = 0$, $-f(a) = \dfrac{f(b) - f(a)}{b - a}(x - a)$

$$(x - a) = \frac{-(b - a) f(a)}{f(b) - f(a)} \quad \text{or} \quad x = a + \frac{(a - b) f(a)}{f(b) - f(a)}$$

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

Repeat the above rule.

Example 11. *Find an approximate value of the root of the equation $x^3 + x - 1 = 0$ near $x = 1$, using the method of false position (regula falsi) two times.*

Solution. $f(x) = x^3 + x - 1 = 0$

$$f(1) = 1 + 1 - 1 = +1$$

$$f(.5) = (.5)^3 + (.5) - 1 = -.375$$

The root lies between .5 and 1.

Let $x_1 = 0.5$ and $x_2 = 1$

$$x = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} \quad \text{or} \quad x = \frac{0.5 f(1) - 1 f(0.5)}{f(1) - f(0.5)}$$

$$= \frac{0.5\,(1) - 1\,(-0.375)}{1 + 0.375} = 0.64$$

Now $f(0.64) = -0.0979$ and $f(1) = 1$

$\therefore$ Root lies between .64 and 1.

$$x_1 = 0.64, \; x_2 = 1$$

$$x = \frac{0.64 f(1) - 1 f(0.64)}{f(1) - f(0.64)} = \frac{0.64\,(1) - 1\,(-0.0979)}{1 + 0.0979} = 0.672$$

Now $f(0.672) = -0.0245$ and $f(1) = 1$

$$x_1 = 0.672 \text{ and } x_2 = 1$$

$$x = \frac{0.672 f(1) - 1 f(0.672)}{f(1) - f(0.672)} = \frac{0.672 + 0.0245}{1 + 0.0245} = 0.6822 \quad \textbf{Ans.}$$

Example 12. *Find the root of the equation $2x - \log_{10} x = 7$ which lies between 3.5 and 4, correct to five places of decimal, using method of false position.*

Solution. $2x - \log_{10} x = 7$ or $2x - \log_{10} x - 7 = 0$

$$f(x) = 2x - \log_{10} x - 7$$

$$f(4) = 8 - \log_{10} 4 - 7 = 1 - 0.60206 = 0.39794$$

$$f(3.5) = 7 - \log_{10} 3.5 - 7 = -0.54407$$

The root x_3 lies between 3.5 and 4.

By False position method

$$x_3 = \frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = \frac{3.5 f(4) - 4 f(3.5)}{f(4) - f(3.5)}$$

$$= \frac{3.5\,(0.39794) - 4\,(-0.54407)}{0.39794 - (-0.54407)} = \frac{1.39279 + 2.17628}{0.94201} = \frac{3.56907}{0.94201} = 3.78878$$

$f(3.78878) = 7.57756 - 0.57850 - 7 = -0.00104$

Again applying False position method

$$x_4 = \frac{3.78878 f(4) - 4 f(3.78878)}{f(4) - f(3.78878)} = \frac{3.78878 \times 0.39794 - 4 \times (-0.00104)}{0.39794 - (-0.001047)}$$

$$= \frac{1.50771 + 0.00416}{0.39898} = \frac{1.51187}{0.39898} = 3.78934 \qquad \textbf{Ans.}$$

Example 13. *Find by the method of Regula Falsi a root of the equation $x^3 + x^2 - 3x - 3 = 0$ lying between 1 and 2.*

Solution. $f(x) = x^3 + x^2 - 3x - 3 = 0$

$$f(1) = 1 + 1 - 3 - 3 = -4 = -ve$$

$$f(2) = 8 + 4 - 6 - 3 = +3 = +ve$$

The root lies between 1 and 2 as $f(1)$ is –ve and $f(2)$ is +ve.

By Regula Falsi method:

$$x_1 = \frac{1 f(2) - 2 f(1)}{f(2) - f(1)} = \frac{1 \times 3 - 2 \times -4}{3 - (-4)} = \frac{11}{7} = 1.571$$

$$f(1.571) = (1.571)^3 + (1.571)^2 - 3\,(1.571) - 3$$

$$= 3.877 + 2.468 - 4.713 - 3 = -1.368 = -ve$$

The root lies between 1.571 and 2 as $f(1.571)$ is –ve and $f(2)$ is + ve.

$$x_2 = \frac{1.571 f(2) - 2 f(1.571)}{f(2) - f(1.571)}$$

$$= \frac{1.571 \times 3 - 2 \times (-1.368)}{3 - (-1.368)} = \frac{4.713 + 2.736}{4.368} = 1.705$$

$$f(1.705) = (1.705)^3 + (1.705)^2 - 3\,(1.705) - 3 = 4.960 + 2.908 - 5.115 - 3$$

$$= -0.247 = -ve.$$

The root lies between 1.705 and 2 as $f(1.705)$ is –ve and $f(2)$ is +ve.

$$x_3 = \frac{1.705 f(2) - 2 f(1.705)}{f(2) - f(1.705)} = \frac{1.705 \times 3 - 2 \times (-0.247)}{3 - (-0.247)} = 1.727$$

Ans.

Example 14. *Find the approximate value, correct to three places of decimals, of the real root which lies between -2 and -3 of the equation $x^3-3x+4=0$, using the method of false position three times in succession.*

Solution. $f(x)=x^3-3x+4=0$

$$x_1=-2,\ x_2=-3$$

$$f(x_1)=f(-2)=(-2)^3-3(-2)+4=-8+6+4=2.$$

$$f(x_2)=f(-3)=(-3)^3-3(-3)+4=-27+9+4=-14$$

$$x=\frac{x_1 f(x_2)-x_2 f(x_1)}{f(x_2)-f(x_1)}=\frac{-2f(-3)-(-3)f(-2)}{f(-3)-f(-2)}=\frac{-2(-14)-(-3)(2)}{(-14)-(2)}=\frac{28+6}{-16}=\frac{34}{-16}$$

$$=-2.125$$

$$f(-2.125)=(-2.125)^3-3(-2.125)+4=-9.596+6.375+4=+0.779$$

$$f(-3)=-14 \text{ and } f(-2.125)=+0.779$$

$\therefore$ Root lies between -2.125 and -3.

$$x=\frac{(-2.125)f(-3)-(-3)f(-2.125)}{f(-3)-f(-2.125)}=\frac{(-2.125)(-14)-(-3)(0.779)}{(-14)-(0.779)}$$

$$=\frac{29.750+2.337}{-14.779}=\frac{32.087}{-14.779}=-2.171$$

$$f(-2.171)=(-2.171)^3-3(-2.171)+4=-10.22+6.513+4=+.293.$$

$$f(-3)=-14 \text{ and } f(-2.171)=+0.293$$

$\therefore$ Root lies between -3 and -2.171.

$$x=\frac{(-2.171)f(-3)-(-3)f(-2.171)}{f(-3)-f(-2.171)}=\frac{(-2.171)(-14)-(-3)(.293)}{-14-.293}$$

$$=\frac{30.494+.879}{-14.293}=\frac{31.273}{-14.293}=-2.188$$ **Ans.**

Example 15. *The negative root of the equation $3x^3+8x^2+8x+5=0$ is to be determined. Find the root by Regula Falsi method. Stop iteration when $f(x_2)<0.02$*

(*A.M.I.E., Summer 2001*)

Solution. $f(x)=3x^3+8x^2+8x+5=0$

$$f(-1)=-3+8-8+5=+2$$

$$f(-1.5)=3(-1.5)^3+8(-1.5)^2+8(-1.5)+5=-10.125+18-12+5=+0.875$$

$$f(-1.6)=3(-1.6)^3+8(-1.6)^2+8(-1.6)+5=-12.288+20.48-12.8+5=+0.392$$

$$f(-1.7)=3(-1.7)^3+8(-1.7)^2+8(-1.7)+5=-14.739+23.12-13.6+5=-0.219$$

Since $f(-1.6)$ and $f(-1.7)$ are of opposite signs so the root lies between -1.6 and -1.7.

By Regula Falsi method:

$$a_1=\frac{-1.6f(-1.7)-(-1.7)f(-1.6)}{f(-1.7)-f(-1.6)}=\frac{-1.6(-0.219)-(-1.7)(0.392)}{-0.219-(0.392)}$$

$$=\frac{0.3504+0.6664}{-.611}=-\frac{1.0168}{.611}=-1.664$$

$$f(-1.664)=3(-1.664)^3+8(-1.664)^2+8(-1.664)+5$$

$$= -13.822 + 22.151 - 13.312 + 5 = 0.017$$

$$f(-1.664) = 0.017 < 0.02$$

Hence the negative root of the given equation is -1.664 **Ans.**

Example 16. *Determine the root of*

$$x^4 + x^3 - 7x^2 - x + 5 = 0$$

which lies between 2 and 3 correct to three decimal places.

Solution.
$$f(x) = x^4 + x^3 - 7x^2 - x + 5 = 0$$
$$f(2) = 16 + 8 - 28 - 2 + 5 = -1$$
$$f(3) = 81 + 27 - 63 - 3 + 5 = +47$$
$$f(2) = -1 \text{ is nearer to zero than } +47.$$

$\therefore$ Root is near to 2.

Let us try on 2.1.

$f(2.1) = (2.1)^4 + (2.1)^3 - 7(2.1)^2 - (2.1) + 5 = 19.4481 + 9.261 - 30.87 - 2.1 + 5 = +0.7391.$

Now the root lies between 2 and 2.1.

By the method of False position :

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{2 f(2.1) - 2.1 f(2)}{f(2.1) - f(2)} = \frac{2(0.739) - 2.1(-1)}{(0.739) - (-1)}$$

$$= \frac{1.4782 + 2.1}{1.739} = \frac{3.5782}{1.739} = 2.0576$$

$$f(2.0576) = (2.0576)^4 + (2.0576)^3 - 7(2.0576)^2 - (2.0576) + 5$$
$$= 17.9244 + 8.7113 - 29.6360 - 2.0576 + 5 = -0.0579$$

$$x_2 = \frac{2.0576 f(2.1) - 2.1 f(2.0576)}{f(2.1) - f(2.0576)} = \frac{2.0576(0.7391) - 2.1(-0.0579)}{0.7391 - (-0.0579)}$$

$$= \frac{1.5208 + 0.1216}{0.7970} = \frac{1.6424}{0.7970} = 2.0607$$

$$f(2.0607) = (2.0607)^4 + (2.0607)^3 - 7(2.0607)^2 - (2.0607) + 5$$
$$= 18.0326 + 8.7507 - 29.7254 - 2.0607 + 5 = -0.0028.$$

$$x_3 = \frac{2.0607 f(2.1) - 2.1 f(2.0607)}{f(2.1) - f(2.0607)} = \frac{2.0607(0.7391) - 2.1(-0.0028)}{0.7391 - (-0.0028)}$$

$$= \frac{1.5231 + 0.0059}{0.7419} = \frac{1.5290}{0.7419} = 2.0609.$$

The root of the given equation is 2.0609. **Ans.**

Exercise 15.3

Solve the following equations by Regula Falsi method :

1. $x^3 - 2x - 5 = 0$ **Ans.** 2.0946
2. $x^3 - 10x^2 + 40x - 35 = 0$ **Ans.** 1.1975.
3. $x^3 + x^2 + 3x + 4 = 0$ **Ans.** -1.22248
4. $x^6 - x^4 - x^3 - 1 = 0$ **Ans.** 1.4036
5. $x^3 - 9x + 1 = 0$ (Root between 2 & 3) **Ans.** 2.9416
6. $x^3 - 5x - 7 = 0$ **Ans.** 2.746
7. $x^3 - x - 1 = 0$ **Ans.** 1.315
8. $3x^3 - 5x^2 + 3x - 5 = 0$ **Ans.** 1.6629

9. The smallest positive root of the equation $x = e^{-x}$ is to be determined. Show that the root lies in (0, 1). Using the Regula Falsi method, find the root correct to three decimals. **Ans.** 0.6065

10. Obtain a root of the equation $x^3 - 4x - 9 = 0$, correct to three decimal places using the method of false position. **Ans.** 2.7064

11. Use the method of false position to find the root of the equation $x^3 - 18 = 0$, given that it lies between 2 and 3. Write down three steps of the procedure. **Ans.** 2.621

12. Find the root of the equation $\tan x + \tanh x = 0$ which lies in the interval (1.6, 3.0) correct to four significant digits using any one of the numerical methods. **Ans.** 2.365 app.

15.5 ITERATION METHOD

Let $$f(x) = 0 \qquad ...(1)$$

(1) can be written as $x = \phi(x)$ ...(2)

where $|\phi'(x)| < 1$ $\therefore$ Let first approximate root be $x_1 = a$

Second Approximation x_2

Putting $x = x_1$ in R.H.S. of (2), we have $x_2 = \phi(x_1)$

Similarly $x_3 = \phi(x_2)$

By repeating this method, we get the better approximation of the root.

Example 17. *Use the method of iteration to solve the equation $x = exp(-x)$, starting with $x = 1.00$. Perform four iterations, taking the readings upto four decimal places.*

Solution. $x = e^{-x}$ ...(1)

$$\phi(x) = e^{-x},\ \phi'(x) = -e^{-x} \qquad |\phi'(x)| = e^{-x}$$

$$|\phi'(1)| = e^{-1} = \frac{1}{e} = .3679 < 1$$

Putting $x = 1$ in (1) we get $x_1 = e^{-1} = 0.3679$

Putting $x = 0.3679$ in (1) we have $x_2 = e^{-0.3679} = 0.692$

Putting $x = 0.692$ in (1), we obtain $x_3 = e^{-0.692} = 0.5$

Putting $x = 0.5$ in (1), we get $x_4 = e^{-0.5} = 0.6065$ **Ans.**

Example 18. *Find a real root of the equation $x^3 + x^2 - 1 = 0$ by the method of iteration.*

Solution. $$f(x) = x^3 + x^2 - 1 \qquad ...(1)$$

$$f(0.7) = 0.343 + 0.49 - 1 = -0.167 = -ve$$

$$f(0.8) = 0.512 + 0.64 - 1 = +0.152 = +ve$$

As $f(0.7)$ and $f(0.8)$ are of opposite sign, so that root lies between 0.7 and 0.8. Let the first approximate root be 0.7.

$$x^3 + x^2 - 1 = 0 \quad \text{or} \quad x^3 = 1 - x^2$$

$$x = (1 - x^2)^{1/3} \qquad x = \phi(x) \qquad \text{where } \phi(x) = (1 - x^2)^{1/3}$$

$$|\phi'(.7)| = .73 < 1$$

$$x_1 = [1 - (.7)^2]^{1/3} = (1 - .49)^{1/3} = (0.51)^{1/3} = 0.799$$

$$x_2 = [1 - (0.799)^2]^{1/3} = (1 - 0.63840)^{1/3} = (0.316)^{1/3} = 0.712$$