Chapter Three
Derivatives

Definition: Let $y = f(x)$ be a function of x . If the limit :

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to o} \frac{\Delta y}{\Delta x}$$

exists and is finite , we call this limit the derivative of f at x and say that f is differentiable at x .

EX-1 **– Find the derivative of the function :** $f(x) = \dfrac{1}{\sqrt{2x+3}}$

***Sol.*:**

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{\sqrt{2(x+\Delta x)+3}} - \dfrac{1}{\sqrt{2x+3}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{2x+3} - \sqrt{2(x+\Delta x)+3}}{\Delta x . \sqrt{2(x+\Delta x)+3}\sqrt{2x+3}} . \frac{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+3) - (2(x+\Delta x)+3)}{\Delta x . \sqrt{2(x+\Delta x)+3}\sqrt{2x+3}(\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3})}$$

$$= \frac{-2}{(2x+3)(\sqrt{2x+3} + \sqrt{2x+3})} = -\frac{1}{\sqrt{(2x+3)^3}}$$

Rules of derivatives : Let c and n are constants, u , v and w are differentiable functions of x :

1. $\dfrac{d}{dx} c = 0$
2. $\dfrac{d}{dx} u^n = n u^{n-1} \dfrac{du}{dx} \Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{u}\right) = -\dfrac{1}{u^2}\dfrac{du}{dx}$
3. $\dfrac{d}{dx} cu = c\dfrac{du}{dx}$
4. $\dfrac{d}{dx}(u \mp v) = \dfrac{du}{dx} \mp \dfrac{dv}{dx}$; $\dfrac{d}{dx}(u \mp v \mp w) = \dfrac{du}{dx} \mp \dfrac{dv}{dx} \mp \dfrac{dw}{dx}$

5. $$\frac{d}{dx}(u.v)=u.\frac{dv}{dx}+v\frac{du}{dx}$$

and $$\frac{d}{dx}(u.v.w)=u.v\frac{dw}{dx}+u.w\frac{dv}{dx}+v.w\frac{du}{dx}$$

6. $$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2} \qquad where \quad v\neq 0$$

EX-2- Find $\frac{dy}{dx}$ for the following functions :

$a)\ y=(x^2+1)^5$ $\qquad$ $b)\ y=[(5-x)(4-2x)]^2$

$c)\ y=(2x^3-3x^2+6x)^{-5}$ $\qquad$ $d)\ y=\frac{12}{x}-\frac{4}{x^3}+\frac{3}{x^4}$

$e)\ y=\frac{(x^2+x)(x^2-x+1)}{x^3}$ $\qquad$ $f)\ y=\frac{x^2-1}{x^2+x-2}$

Sol.-

$a)\ \frac{dy}{dx}=5(x^2+1)^4.2x=10x(x^2+1)^4$

$b)\ \frac{dy}{dx}=2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$

$=8(5-x)(2-x)(2x-7)$

$c)\ \frac{dy}{dx}=-5(2x^3-3x^2+6x)^{-6}(6x^2-6x+6)$

$=-30(2x^3-3x^2+6x)^{-6}(x^2-x+1)$

$d)\ y=12x^{-1}-4x^{-3}+3x^{-4}\Rightarrow\frac{dy}{dx}=-12x^{-2}+12x^{-4}-12x^{-5}$

$\Rightarrow\frac{dy}{dx}=-\frac{12}{x^2}+\frac{12}{x^4}-\frac{12}{x^5}$

$e)\ y=\frac{(x+1)(x^2-x+1)}{x^3}\Rightarrow$

$$\frac{dy}{dx}=\frac{x^3[(x^2-x+1)+(x+1)(2x-1)]-3x^2(x+1)(x^2-x+1)}{x^6}=-\frac{3}{x^4}$$

$$f)\ \frac{dy}{dx}=\frac{2x(x^2+x-2)-(x^2-1)(2x+1)}{(x^2+x-2)^2}=\frac{x^2-2x+1}{(x^2+x-2)^2}$$

The Chain Rule:

1. **Suppose that $h = g_o f$ is the composite of the differentiable functions $y = g(t)$ and $x = f(t)$, then h is a differentiable functions of x whose derivative at each value of x is :**

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

2. **If y is a differentiable function of t and t is differentiable function of x, then y is a differentiable function of x :**

$$y = g(t) \;\; and \;\; t = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$$

EX-3 **– Use the chain rule to express dy/dx in terms of x and y :**

$$a) \quad y = \frac{t^2}{t^2+1} \qquad and \quad t = \sqrt{2x+1}$$

$$b) \quad y = \frac{1}{t^2+1} \qquad and \quad x = \sqrt{4t+1}$$

$$c) \quad y = \left(\frac{t-1}{t+1}\right)^2 \qquad and \quad x = \frac{1}{t^2} - 1 \quad at \quad t = 2$$

$$d) \quad y = 1 - \frac{1}{t} \qquad and \quad t = \frac{1}{1-x} \quad at \quad x = 2$$

Sol.-

$$a) \quad y = \frac{t^2}{t^2+1} \Rightarrow \frac{dy}{dt} = \frac{2t(t^2+1) - 2t.t^2}{(t^2+1)^2} = \frac{2t}{(t^2+1)^2}$$

$$t = (2x+1)^{\frac{1}{2}} \Rightarrow \frac{dt}{dx} = \frac{1}{2}.(2x+1)^{-\frac{1}{2}}.2 = \frac{1}{\sqrt{2x+1}}$$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{2t}{(t^2+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{2\sqrt{2x+1}}{((2x+1)+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)^2}$$

$$b)\quad y=(t^2+1)^{-1}\Rightarrow \frac{dy}{dx}=-2t(t^2+1)^{-2}=-\frac{2t}{(t^2+1)^2}$$

$$x=(4t+1)^{\frac{1}{2}}\Rightarrow \frac{dx}{dt}=\frac{1}{2}(4t+1)^{-\frac{1}{2}}.4=\frac{2}{\sqrt{4t+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{2t}{(t^2+1)^2}\div\frac{2}{\sqrt{4t+1}}=-\frac{t\sqrt{4t+1}}{(t^2+1)^2}$$

$$=-\frac{x^2-1}{4}.x\div\frac{1}{y^2}=-\frac{xy^2(x^2-1)}{4}$$

$$\text{where}\quad x=\sqrt{4t+1}\Rightarrow t=\frac{x^2-1}{4}$$

$$\text{where}\quad y=\frac{1}{t^2+1}\Rightarrow t^2+1=\frac{1}{y}$$

$$c)\quad y=\left(\frac{t-1}{t+1}\right)^2\Rightarrow \frac{dy}{dt}=2\left(\frac{t-1}{t+1}\right)\frac{t+1-(t-1)}{(t+1)^2}=\frac{4(t-1)}{(t+1)^3}$$

$$\Rightarrow\left[\frac{dy}{dt}\right]_{t=2}=\frac{4(2-1)}{(2+1)^3}=\frac{4}{27}$$

$$x=\frac{1}{t^2}-1\Rightarrow \frac{dx}{dt}=-\frac{2}{t^3}\Rightarrow\left[\frac{dx}{dt}\right]_{t=2}=-\frac{2}{2^3}=-\frac{1}{4}$$

$$\left[\frac{dy}{dx}\right]_{t=2}=\left[\frac{dy}{dt}\div\frac{dx}{dt}\right]_{t=2}=\frac{4}{27}\div\left(-\frac{1}{4}\right)=-\frac{16}{27}$$

$$d)\quad t=\frac{1}{1-x}=\frac{1}{1-2}=-1\qquad at\quad x=2$$

$$y=1-\frac{1}{t}\Rightarrow\frac{dy}{dt}=\frac{1}{t^2}\Rightarrow\left[\frac{dy}{dt}\right]_{t=-1}=\frac{1}{(-1)^2}=1$$

$$t=(1-x)^{-1}\Rightarrow\frac{dt}{dx}=-(1-x)^{-2}(-1)=\frac{1}{(1-x)^2}$$

$$\Rightarrow\left[\frac{dt}{dx}\right]_{x=2}=\frac{1}{(1-2)^2}=1$$

$$\left[\frac{dy}{dx}\right]_{x=2}=\left[\frac{dy}{dt}\right]_{x=2}.\left[\frac{dt}{dx}\right]_{x=2}=1*1=1$$

<u>*Higher derivatives*</u> *:* If a function $y = f(x)$ possesses a derivative at every point of some interval , we may form the function $f'(x)$ and talk about

its derivate , if it has one . The procedure is formally identical with that used before , that is :

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}f'(x)=\lim_{\Delta x\to 0}\frac{f'(x+\Delta x)-f'(x)}{\Delta x}$$

if the limit exists .

This derivative is called the second derivative of y with respect to x . It is written in a number of ways , for example,

$$y''\,,\ f''(x)\,,\ or\ \frac{d^2f(x)}{dx^2}\,.$$

In the same manner we may define third and higher derivatives , using similar notations . The *nth* derivative may be written :

$$y^{(n)},\ f^{(n)}(x),\ \frac{d^ny}{dx^n}\ .$$

***EX-4*- Find all derivatives of the following function :**

$$y = 3x^3 - 4x^2 + 7x + 10$$

***Sol.*-**

$$\frac{dy}{dx}=9x^2-8x+7 \quad , \quad \frac{d^2y}{dx^2}=18x-8$$

$$\frac{d^3y}{dx^3}=18 \quad , \quad \frac{d^4y}{dx^4}=0=\frac{d^5y}{dx^5}=....$$

***Ex-5* – Find the third derivative of the following function :**

$$y=\frac{1}{x}+\sqrt{x^3}$$

***Sol.*-**

$$\frac{dy}{dx}=-\frac{1}{x^2}+\frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{2}{x^3}+\frac{3}{4}x^{-\frac{1}{2}}$$

$$\frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8}x^{-\frac{3}{2}} \qquad \Rightarrow \frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8\sqrt{x^3}}$$

***Implicit Differentiation*:** **If the formula for f is an algebraic combination of powers of x and y . To calculate the derivatives of these implicitly defined functions , we simply differentiate both sides of the defining equation with respect to x .**

***EX-6*- Find $\frac{dy}{dx}$ for the following functions:**

$a)\ x^2 . y^2 = x^2 + y^2$

$b)\ (x+y)^3 + (x-y)^3 = x^4 + y^4$

$c)\ \frac{x-y}{x-2y} = 2 \text{ at } P(3,1)$

$d)\ xy + 2x - 5y = 2 \text{ at } P(3,2)$

Sol.

$$a)\ x^2(2y\frac{dy}{dx}) + y^2(2x) = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2 y - y}$$

$$b)\ 3(x+y)^2(1+\frac{dy}{dx}) + 3(x-y)^2(1-\frac{dy}{dx}) = 4x^3 + 4y^3\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

$$c)\ \frac{(x-2y)(1-\frac{dy}{dx}) - (x-y)(1-2\frac{dy}{dx})}{(x-2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,1)} = \frac{1}{3}$$

$$d)\ x\frac{dy}{dx} + y + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{5-x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,2)} = \frac{2+2}{5-3} = 2$$

Trigonometric functions **: If *u* is any differentiable function of *x* , then :**

$$9)\ \frac{d}{dx}\sin u = \cos u.\frac{du}{dx}$$

$$10)\ \frac{d}{dx}\cos u = -\sin u.\frac{du}{dx}$$

$$11)\ \frac{d}{dx}\tan u = \sec^2 u.\frac{du}{dx}$$

$$12)\ \frac{d}{dx}\cot u = -\csc^2 u.\frac{du}{dx}$$

$$13)\ \frac{d}{dx}\sec u = \sec u.\tan u.\frac{du}{dx}$$

$$14)\ \frac{d}{dx}\csc u = -\csc u.\cot u.\frac{du}{dx}$$

EX-7- **Find $\frac{dy}{dx}$ for the following functions :**

a) $y = \tan(3x^2)$ b) $y = (\csc x + \cot x)^2$

c) $y = 2\sin\frac{x}{2} - x\mathrm{Cos}\frac{x}{2}$ d) $y = \tan^2(\cos x)$

e) $x + \tan(xy) = 0$ f) $y = \sec^4 x - \tan^4 x$

Sol.-

$$a)\ \frac{dy}{dx}=sec^2(3x^2).6x=6x.sec^2(3x^2)$$

$$b)\ \frac{dy}{dx}=2(csc\,x+cot\,x)(-csc\,x.cot\,x-csc^2\,x)=-2csc\,x.(csc\,x+cot\,x)^2$$

$$c)\ \frac{dy}{dx}=2cos\frac{x}{2}.\frac{1}{2}-\left[x(-sin\frac{x}{2}).\frac{1}{2}+cos\frac{x}{2}\right]=\frac{x}{2}.sin\frac{x}{2}$$

$$d)\ \frac{dy}{dx}=2.tan(cos\,x).sec^2(cos\,x).(-sin\,x)=-2.sin\,x.tan(cos\,x).sec^2(cos\,x)$$

$$e)\ 1+sec^2(xy).(x\frac{dy}{dx}+y)=0\Rightarrow\frac{dy}{dx}=-\frac{1+y.sec^2(xy)}{x.sec^2(xy)}=-\frac{cos^2(xy)+y}{x}$$

$$f)\ \frac{dy}{dx}=4sec^3\,x.sec\,x.tan\,x-4.tan^3\,x.sec^2\,x=4\,tan\,x.sec^2\,x$$

<u>EX-8</u>- **Prove that :**

$$a)\ \frac{d}{dx}tan\,u=sec^2\,u.\frac{du}{dx}\qquad b)\ \frac{d}{dx}sec\,u=sec\,u.tan\,u.\frac{du}{dx}$$

<u>Proof</u> :

$$a)\quad L.H.S.=\frac{d}{dx}tan\,u=\frac{d}{dx}\frac{sin\,u}{cos\,u}=\frac{cos\,u.cos\,u.\frac{du}{dx}-sin\,u.(-sin\,u)\frac{du}{dx}}{cos^2\,u}$$

$$=\frac{cos^2\,u+sin^2\,u}{cos^2\,u}.\frac{du}{dx}=\frac{1}{cos^2\,u}.\frac{du}{dx}=sec^2\,u.\frac{du}{dx}=R.H.S.$$

$$b)\quad L.H.S.=\frac{d}{dx}sec\,u=\frac{d}{dx}\frac{1}{cos\,u}=-\frac{1}{cos^2\,u}(-sin\,u)\frac{du}{dx}$$

$$=\frac{1}{cos\,u}.\frac{sin\,u}{cos\,u}.\frac{du}{dx}=sec\,u.tan\,u.\frac{du}{dx}=R.H.S.$$

Problems (3)

1. Find $\frac{dy}{dx}$ for the following functions :

1) $y=(x-3)(1-x)$ $\quad$ $(ans.: 4-2x)$

2) $y=\frac{ax+b}{x}$ $\quad$ $(ans.: -\frac{b}{x^2})$

3) $y=\frac{3x+4}{2x+3}$ $\quad$ $(ans.: \frac{1}{(2x+3)^2})$

4) $y=3x^3-2\sqrt{x}+\frac{5}{x^2}$ $\quad$ $(ans.: 9x^2-\frac{1}{\sqrt{x}}-\frac{10}{x^3})$

5) $y=\left(\sqrt{x^3}-\frac{1}{\sqrt{x^3}}\right)^2$ $\quad$ $(ans.: \frac{3(x^6-1)}{x^4})$

6) $y=(2x-1)^2(3x+2)^3+\frac{1}{(x-2)^2}$ $\quad$ $(ans.: (2x-1)(3x+2)^2(30x-1)-\frac{2}{(x-2)^3})$

7) $y=\mathrm{Sin}x^3$ $\quad$ $(ans.: 3x^2.\mathrm{Cos}x^3)$

8) $y=\mathrm{Cos}^{-3}(5x^2+2)$ $\quad$ $(ans.: \frac{30x.\mathrm{Sin}(5x^2+4)}{\mathrm{Cos}^4(5x^2+4)})$

9) $y=\tan x.\sin x$ $\quad$ $(ans.: \mathrm{Sin}x+\tan x.\mathrm{Sec}x)$

10) $y=\tan(\mathrm{Sec}x)$ $\quad$ $(ans.: \mathrm{Sec}^2(\mathrm{Sec}x).\mathrm{Sec}x.\tan x)$

11) $y=\mathrm{Cot}^3\left(\frac{x+1}{x-1}\right)$ $\quad$ $(ans.: \frac{6}{(x-1)^2}.\mathrm{Cot}^2\left(\frac{x+1}{x-1}\right).\mathrm{Csc}^2\left(\frac{x+1}{x-1}\right))$

12) $y=\frac{\mathrm{Cos}x}{x}$ $\quad$ $(ans.: -\frac{x.\mathrm{Sin}x+\mathrm{Cos}x}{x^2})$

13) $y=\sqrt{\tan\sqrt{2x+7}}$ $\quad$ $(ans.: \frac{\mathrm{Sec}^2\sqrt{2x+7}}{2\sqrt{2x+7}\sqrt{\tan\sqrt{2x+7}}})$

14) $y=x^2.\mathrm{Sin}x$ $\quad$ $(ans.: x^2.\mathrm{Cos}x+2x.\mathrm{Sin}x)$

15) $y=\mathrm{Csc}^{-\frac{2}{3}}\sqrt{5x}$ $\quad$ $(ans.: \frac{5}{3\sqrt{5x}}.\frac{\mathrm{Cot}\sqrt{5x}}{\mathrm{Csc}^{\frac{2}{3}}\sqrt{5x}})$

2. Verify the following derivatives:

$$a)\quad \frac{d}{dx}\left[5x+(\sqrt{x}+\frac{1}{\sqrt{x}})^2\right]=6-\frac{1}{x^2}$$

$$b)\quad \frac{d}{dx}\left[\sqrt{x}(ax^2+bx+c)\right]=\frac{1}{2\sqrt{x}}(5ax^2+3bx+c)$$

3. Find the derivative of y with respect to x in the following functions :

$a)\ y=\dfrac{u^2}{u^2+1}$ *and* $u=3x^3-2$ $\qquad (ans.: \dfrac{18x^2y^2}{(3x^3-2)^3})$

$b)\ y=\sqrt{u}+2u$ *and* $u=x^2-3$ $\qquad (ans.: \dfrac{x}{\sqrt{x^2-3}}+4x)$

4. Find the second derivative for the following functions :

$a)\quad y=(x+\dfrac{1}{x})^3$ $\qquad (ans.: 6x+\dfrac{6}{x^3}+\dfrac{12}{x^5})$

$b)\quad f(x)=\sqrt{2x}+\dfrac{2\sqrt{2}}{\sqrt{x}}$ at $x=2$ $\qquad (ans.: \dfrac{1}{4})$

$c)\quad x^2-2xy+y^2-16x=0$ $\qquad (ans.: \mp x^{-\frac{3}{2}})$

5. Find the third derivative of the function : $y=\sqrt{x^3}$ $\qquad (ans.: -\dfrac{3}{8y})$

6. Show for $y=\dfrac{u}{v}$ **that** $y''=\dfrac{v(vu''-uv'')-2v'(vu'-uv')}{v^3}$.

7. Show for $y = u.v$ **that** $y''' = uv''' + 3u'\,v'' + 3u''\,v' + u'''\,v$.

8. **Show that** $y = 35x^4 - 30x^2 + 3$ **satisfies** $(1 - x^2)y'' - 2xy' + 20y = 0$.

9. **Find** $\dfrac{dy}{dx}$ **for the following implicit functions :**

a) $x^3 + 4x\sqrt{y} - \dfrac{5y^2}{x} = 3$ $\quad$ (ans. : $\dfrac{3x^2 + 5y^2x^{-2} + 4\sqrt{y}}{10x^{-1}y - \dfrac{2x}{\sqrt{y}}}$)

b) $\sqrt{xy} + 1 = y$ $\quad$ (ans. : $\dfrac{y}{2\sqrt{xy} - x}$)

c) $3xy = (x^3 + y^3)^{\frac{3}{2}}$ $\quad$ (ans. : $\dfrac{3x^2\sqrt{x^3 + y^3} - 2y}{2x - 3y^2\sqrt{x^3 + y^3}}$)

f) $y^2.\sin(xy) = \tan x$ $\quad$ (ans. : $\dfrac{\sec^2 x - y^3.\cos(xy)}{2y.\sin(xy) + xy^2.\cos(xy)}$)

Chapter Four
Applications of derivatives

L'Hopital rule :

Suppose that $f(x_o) = g(x_o) = 0$ and that the functions f and g are both differentiable on an open interval (a, b) that contains the point x_o. Suppose also that $g'(x) \neq 0$ at every point in (a, b) except possibly x_o. Then :

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} \quad \text{provided the limit exists .}$$

Differentiate f and g as long as you still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = x_o$. Stop differentiating as soon as you get something else . L'Hopital's rule does not apply when either the numerator or denominator has a finite non-zero limit .

EX-1 **– Evaluate the following limits :**

$$1)\ \lim_{x\to o} \frac{\sin x}{x} \qquad 2)\ \lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-4}$$

$$3)\ \lim_{x\to 0} \frac{x-\sin x}{x^3} \qquad 4)\ \lim_{x\to \frac{\pi}{2}} -\left(x-\frac{\pi}{2}\right).\tan x$$

Sol. –

$$1)\quad \lim_{x\to 0} \frac{\sin x}{x} \Rightarrow \frac{0}{0} \ \text{using L' Hoptal' s rule} \Rightarrow$$

$$= \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$$

$$2)\quad \lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-4} \Rightarrow \frac{0}{0} \ \text{using L' Hoptal' s rule} \Rightarrow$$

$$= \lim_{x\to 2} \frac{\frac{x}{\sqrt{x^2+5}}}{2x} = \lim_{x\to 2} \frac{1}{2\sqrt{x^2+5}} = \frac{1}{2\sqrt{4+5}} = \frac{1}{6}$$

3) $\lim_{x\to 0}\frac{x-\sin x}{x^3} \Rightarrow \frac{0}{0}$ using L' Hoptal' s rule $\Rightarrow$

$$=\lim_{x\to 0}\frac{1-\cos x}{3x^2} \Rightarrow \frac{0}{0} \text{ using L' Hopital' s rule} \Rightarrow$$
$$=\frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x}=\frac{1}{6}$$

4) $\lim_{x\to\frac{\pi}{2}} -(x-\frac{\pi}{2})\tan x \Rightarrow 0.\infty$ we can't using L' Hoptal' s rule $\Rightarrow$

$$=\lim_{x\to\frac{\pi}{2}} -\frac{x-\frac{\pi}{2}}{\cos x}.\lim_{x\to\frac{\pi}{2}}\sin x \Rightarrow \frac{0}{0} \text{ using L' Hopital' s rule} \Rightarrow$$
$$=\lim_{x\to\frac{\pi}{2}} -\frac{1}{-\sin x}.\lim_{x\to\frac{\pi}{2}}\sin x=\frac{1}{\sin\frac{\pi}{2}}.\sin\frac{\pi}{2}=1$$

<u>The slope of the curve</u> :

Secant to the curve is a line through two points on a curve. Slopes and tangent lines :

1. **we start with what we can calculate , namely the slope of secant through *P* and a point *Q* nearby on the curve .**
2. **we find the limiting value of the secant slope (if it exists) as *Q* approaches *p* along the curve .**
3. **we take this number to be the slope of the curve at *P* and define the tangent to the curve at *P* to be the line through *p* with this slope .**

The derivative of the function *f* is the slope of the curve :

$$\text{the slope} = m = f'(x)=\frac{dy}{dx}$$

***<u>EX-2</u>*- Write an equation for the tangent line at *x = 3* of the curve :**

$$f(x)=\frac{1}{\sqrt{2x+3}}$$

***<u>Sol.</u>*-**

$$m=f'(x)=-\frac{1}{\sqrt{(2x+3)^3}} \Rightarrow [m]_{x=3}=f'(3)=-\frac{1}{27}$$
$$f(3)=\frac{1}{\sqrt{2*3+3}}=\frac{1}{3}$$

The equation of the tangent line is :

$$y-\frac{1}{3}=-\frac{1}{27}(x-3)\Rightarrow 27y+x=12$$

<u>Velocity and acceleration and other rates of changes</u> :

- The average velocity of a body moving along a line is :

$$v_{av}=\frac{\Delta s}{\Delta t}=\frac{f(t+\Delta t)-f(t)}{\Delta t}=\frac{displacement}{time\ travelled}$$

The instantaneous velocity of a body moving along a line is the derivative of its position $s=f(t)$ with respect to time t .

i.e. $v=\dfrac{ds}{dt}=\lim\limits_{\Delta t\to 0}\dfrac{\Delta s}{\Delta t}$

- The rate at which the particle's velocity increase is called its acceleration a . If a particle has an initial velocity v and a constant acceleration a, then its velocity after time t is $v+at$.

$$average\ acceleration=a_{av}=\frac{\Delta v}{\Delta t}$$

The acceleration at an instant is the limit of the average acceleration for an interval following that instant , as the interval tends to zero .

i.e. $a=\lim\limits_{\Delta t\to 0}\dfrac{\Delta v}{\Delta t}$

- The average rate of a change in a function $y=f(x)$ over the interval from x to $x+\Delta x$ is :

$$average\ rate\ of\ change=\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

The instantaneous rate of change of f at x is the derivative.

$f'(x)=\lim\limits_{\Delta x\to 0}\dfrac{f(x+\Delta x)-f(x)}{\Delta x}$ provided the limit exists .

<u>EX-3</u>- The position s (in meters) of a moving body as a function of time t (in second) is : $s=2t^2+5t-3$; find :

a) The displacement and average velocity for the time interval from $t = 0$ to $t = 2$ seconds .

b) The body's velocity at $t = 2$ seconds .

Sol.-

$$a)\quad 1)\qquad \Delta s = s(t+\Delta t) - s(t) = 2(t+\Delta t)^2 + 5(t+\Delta t) - 3 - \left[2t^2 + 5t - 3\right]$$
$$= (4t+5)\Delta t + 2(\Delta t)^2$$
$$at\ t = 0\ and\ \Delta t = 2 \Rightarrow \Delta s = (4*0+5)*2 + 2*2^2 = 18$$

$$2)\quad v_{av} = \frac{\Delta s}{\Delta t} = \frac{(4t+5)\Delta t + 2(\Delta t)^2}{\Delta t} = 4t + 5 + 2.\Delta t$$
$$at\ t = 0\ \ and\ \Delta t = 2 \Rightarrow v_{av} = 4*0 + 5 + 2*2 = 9$$

$$b)\quad v(t) = \frac{d}{dt} f(t) = 4t + 5$$
$$v(2) = 4*2 + 5 = 13$$

EX-4- A particle moves along a straight line so that after t (seconds) , its distance from O a fixed point on the line is s (meters) , where $s = t^3 - 3t^2 + 2t$:

i) when is the particle at O ?

ii) what is its velocity and acceleration at these times ?

iii) what is its average velocity during the first second ?

iv) what is its average acceleration between $t = 0$ and $t = 2$?

Sol. –

$$i)\quad at\ \ s = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-1)(t-2) = 0$$
$$either\ \ t = 0\ \ or\ \ t = 1\ \ or\ \ t = 2\ \ \ sec.$$

$$ii)\quad velocity = v(t) = 3t^2 - 6t + 2 \Rightarrow v(0) = 2m/s$$
$$\Rightarrow v(1) = -1m/s$$
$$\Rightarrow v(2) = 2m/s$$

$$acceleration = a(t) = 6t - 6 \Rightarrow a(0) = -6m/s^2$$
$$\Rightarrow a(1) = 0m/s^2$$
$$\Rightarrow a(2) = 6m/s^2$$

$$iii)\quad v_{av}=\frac{\Delta s}{\Delta t}=\frac{s(1)-s(0)}{1-0}=\frac{1-3+2-0}{1}=0m/s$$

$$iv)\quad a_{av}=\frac{\Delta v}{\Delta t}=\frac{v(2)-v(0)}{2-0}=\frac{2-2}{2}=0m/s^2$$

<u>Maxima and Minima</u> :

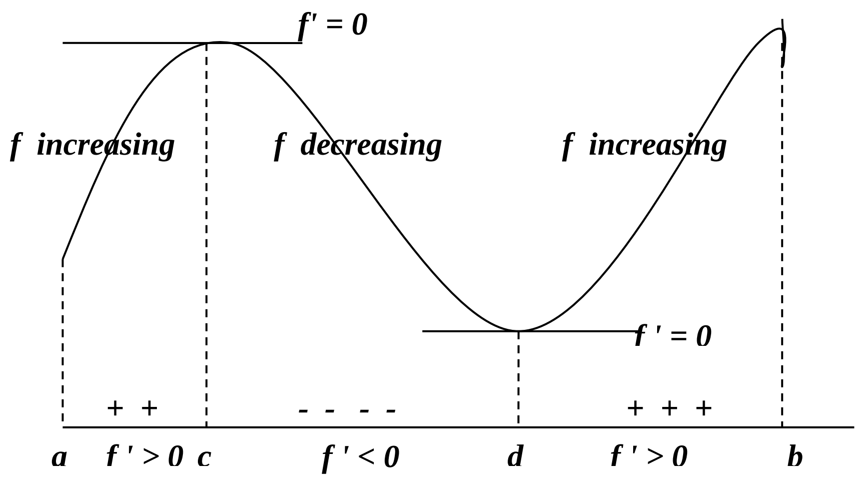
<u>Increasing and decreasing function</u> : **Let f be defined on an interval and x_1 , x_2 denoted a number on that interval :**

- **If $f(x_1) < f(x_2)$ when ever $x_1 < x_2$ then f is increasing on that interval .**
- **If $f(x_1) > f(x_2)$ when ever $x_1 < x_2$ then f is decreasing on that interval .**
- **If $f(x_1) = f(x_2)$ for all values of x_1 , x_2 then f is constant on that interval .**

<u>The first derivative test for rise and fall</u> : **Suppose that a function f has a derivative at every point x of an interval I. Then :**

- **f increases on I if $f'(x)>o,\quad \forall x\in I$**
- **f decreases on I if $f'(x)>o,\quad \forall x\in I$**

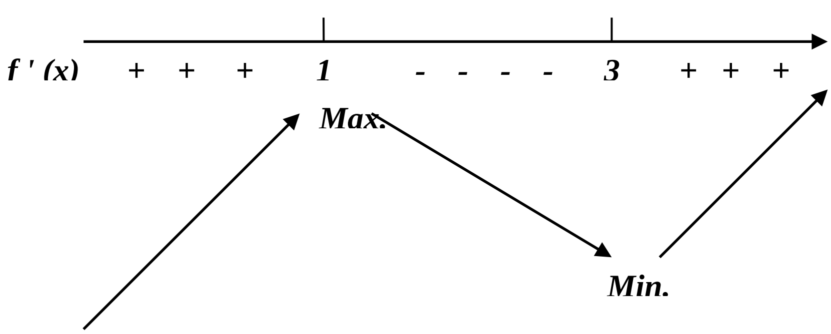
If f' changes from positive to negative values as x passes from left to right through a point c , then the value of f at c is a local maximum value of f , as shown in below figure . That is $f(c)$ is the largest value the function takes in the immediate neighborhood at $x = c$.



Similarly , if *f '* changes from negative to positive values as *x* passes left to right through a point *d* , then the value of *f* at *d* is a local minimum value of *f* . That is *f(d)* is the smallest value of *f* takes in the immediate neighborhood of *d* .

EX-5 **– Graph the function :** $y = f(x) = \frac{x^3}{3} - 2x^2 + 3x + 2$ **.**

Sol.- $f'(x) = x^2 - 4x + 3 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$



The function has a local maximum at *x = 1* and a local minimum at *x = 3* .

To get a more accurate curve , we take :

x	0	1	2	3	4
f(x)	2.3	3.3	2.7	2	3

Then the graph of the function is :

<u>Concave down and concave up</u> : **The graph of a differentiable function $y = f(x)$ is concave down on an interval where f' decreases , and concave up on an interval where f' increases.**

<u>The second derivative test for concavity</u> : **The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$, concave up on any interval where $y'' > 0$.**

<u>Point of inflection</u> : **A point on the curve where the concavity changes is called a point of inflection . Thus , a point of inflection on a twice – differentiable curve is a point where y'' is positive on one side and negative on other , i.e. $y'' = 0$.**

EX-6 **– Sketch the curve :** $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$.

Sol. **-**

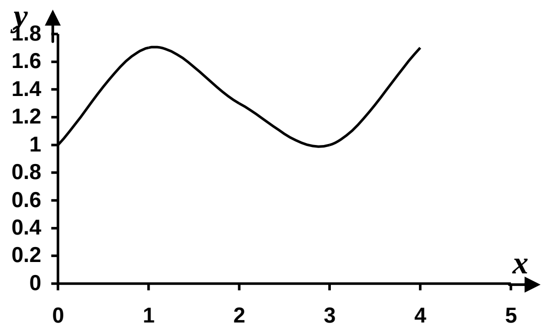
$$y' = \frac{1}{2}x^2 - 2x + \frac{3}{2} = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$$

$$y'' = x - 2 \Rightarrow at\ x = 1 \Rightarrow y'' = 1 - 2 = -1 < 0\ concave\ down .$$

$$\Rightarrow at\ x = 3 \Rightarrow y'' = 3 - 2 > 0 \qquad concave\ up .$$

$$\Rightarrow at\ y'' = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2 \quad point\ of\ inflection .$$

x	*0*	*1*	*2*	*3*	*4*
y	*1*	*1.7*	*1.3*	*1*	*1.7*



EX-7 **– What value of** a **makes the function :**

$f(x) = x^2 + \frac{a}{x}$ **, have :**

i) a local minimum at $x = 2$ **?**
ii) a local minimum at $x = -3$ **?**
iii) a point of inflection at $x = 1$ **?**
iv) show that the function can't have a local maximum for any value of a **.**

Sol. **–**

$$f(x) = x^2 + \frac{a}{x} \Rightarrow \frac{df}{dx} = 2x - \frac{a}{x^2} = 0 \Rightarrow a = 2x^3 \ and \ \frac{d^2y}{dx^2} = 2 + \frac{2a}{x^3}$$

$$i)\quad at\ x=2 \Rightarrow a = 2*8 = 16\ and\ \frac{d^2 f}{dx^2} = 2 + \frac{2*16}{2^3} = 6 > 0\ Mini.$$

$$ii)\quad at\ x=-3 \Rightarrow a = 2(-3)^3 = -54\ and\ \frac{d^2 f}{dx^2} = 2 + \frac{2(-54)}{(-3)^3} = 6 > 0\ Mini.$$

$$iii)\quad at\ x=1 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2a}{1} = 0 \Rightarrow a = -1$$

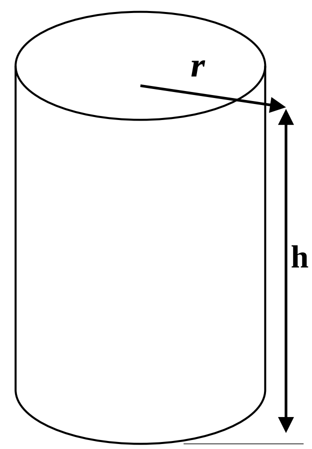
$$iv)\quad a = 2x^3 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2(2x^3)}{x^3} = 6 > 0$$

Since $\frac{d^2 f}{dx^2} > 0$ *for all value of* x *in* $a = 2x^3$.

Hence the function don't have a local maximum .

<u>*EX-8*</u> – What are the best dimensions (use the least material) for a tin can which is to be in the form of a right circular cylinder and is to hold *1* gallon (*231* cubic inches) ?

<u>*Sol.*</u> – The volume of the can is :



$$v = \pi r^2 h = 231 \Rightarrow h = \frac{231}{\pi r^2}$$

where *r* is radius , *h* is height .

The total area of the outer surface (top, bottom , and side) is :

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{231}{\pi r^2} \Rightarrow A = 2\pi r^2 + \frac{462}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{462}{r^2} = 0 \Rightarrow r = 3.3252 \text{ inches}$$

T

$$\frac{d^2 A}{dr^2} = 4\pi + \frac{924}{r^3} = 4\pi + \frac{924}{(3.3252)^3} = 37.714 > 0 \Rightarrow min.$$

$$h = \frac{231}{\pi r^2} = \frac{231}{\frac{22}{7}(3.3252)^2} = 6.6474 \text{ inches}$$

he dimensions of the can of volume *1* gallon have minimum surface area are :

***r = 3.3252* in. and *h = 6.6474* in.**

***<u>EX-9</u>* – A wire of length *L* is cut into two pieces , one being bent to form a square and the other to form an equilateral triangle . How should the wire be cut :**

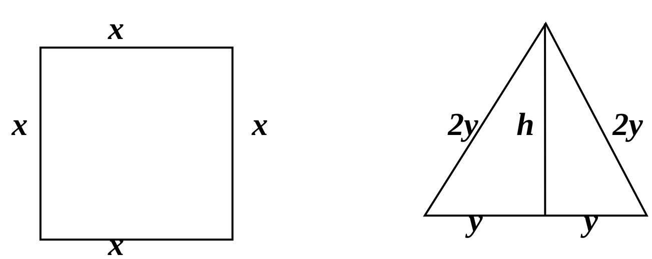
a) if the sum of the two areas is minimum.

b) if the sum of the two areas is maximum.

c)

***<u>Sol.</u> :* Let *x* is a length of square.**

***2y* is the edge of triangle .**



The perimeter is $p = 4x + 6y = L \Rightarrow x = \frac{1}{4}(L - 6y)$.

$(2y)^2 = y^2 + h^2 \Rightarrow h = \sqrt{3}\,y$ *from triangle* .

The total area is $A = x^2 + yh = \frac{1}{16}(L - 6y)^2 + y\sqrt{3}\,y$

$$\Rightarrow A = \frac{1}{16}(L - 6y)^2 + \sqrt{3}\,y^2$$

$$\frac{dA}{dy}=\frac{-3}{4}(L-6y)+2\sqrt{3}\,y=0\Rightarrow y=\frac{3L}{18+8\sqrt{3}}$$
$$\frac{d^2A}{dy^2}=\frac{9}{2}+2\sqrt{3}>0\Rightarrow min.$$

a) **To minimized total areas cut for triangle** $6y=\dfrac{9L}{9+4\sqrt{3}}$

And for square $L-\dfrac{9L}{9+4\sqrt{3}}=\dfrac{4\sqrt{3}L}{9+4\sqrt{3}}$ **.**

b) **To maximized the value of** A **on endpoints of the interval**

$$0\le 4x\le L\Rightarrow 0\le x\le\frac{L}{4}$$

$$at\ x=0\Rightarrow y=\frac{L}{6}\ and\ h=\frac{L}{2\sqrt{3}}\Rightarrow A_1=\frac{L^2}{12\sqrt{3}}$$
$$at\ \ x=\frac{L}{4}\Rightarrow y=0\Rightarrow A_2=\frac{L^2}{16}$$

$$Since\ \ A_2=\frac{L^2}{16}>A_1=\frac{L^2}{12\sqrt{3}}$$

Hence the wire should not be cut at all but should be bent into a square .

Problems (4)

1. Find the velocity v if a particle's position at time t is $s = 180t - 16t^2$ When does the velocity vanish ? ***(ans.: 5.625)***

2. If a ball is thrown straight up with a velocity of 32 ft./sec. , its high after t sec. is given by the equation $s = 32t - 16t^2$. At what instant will the ball be at its highest point ? and how high will it rise ?
(ans.: 1, 16)

3. A stone is thrown vertically upwards at 35 m./sec. . Its height is : $s = 35t - 4.9t^2$ in meter above the point of projection where t is time in second later :
a) What is the distance moved, and the average velocity during the 3^{rd} sec. (from $t = 2$ to $t = 3$) ?
b) Find the average velocity for the intervals $t = 2$ to $t = 2.5$, $t = 2$ to $t = 2.1$; $t = 2$ to $t = 2 + h$.
c) Deduce the actual velocity at the end of the 2^{nd} sec. .
(ans.: a) 10.5 , 10.5 ; b) 12.95, 14.91, 15.4-4.9h , c) 15.4)

4. A stone is thrown vertically upwards at 24.5 m./sec. from a point on the level with but just beyond a cliff ledge . Its height above the ledge t sec. later is $4.9t (5 - t)$ m. . If its velocity is v m./sec. , differentiate to find v in terms of t :
i) when is the stone at the ledge level ?
ii) find its height and velocity after 1 , 2 , 3 , and 6 sec. .
iii) what meaning is attached to negative value of s ? a negative value of v ?
iv) when is the stone momentarily at rest ? what is the greatest height reached ?
v) find the total distance moved during the 3^{rd} sec. .
(ans.:v=24.5-9.8t; i)0,5; ii)19.6,29.4,29.4,-29.4;14.7,4.9, -4.9,-34.3; iv)2.5;30.625; v)2.45)

5. A stone is thrown vertically downwards with a velocity of 10 m./sec. , and gravity produces on it an acceleration of 9.8 m./sec.^{2} :
a) what is the velocity after 1 , 2 , 3 , t sec. ?

b) sketch the velocity –time graph . ***(ans.: 19.8, 29.6, 39.4,10+9.8t)***

6. A car accelerates from *5* km./h. to *41* km./h. in *10* sec. . Express this acceleration in : i)km./h. per sec. ii) m./sec.^{2}, iii) km./h.^{2} .
(ans.: i)3.6; ii)1; iii) 12960)

7. A car can accelerate at *4* m./sec.^{2} . How long will it take to reach *90* km./h. from rest ? ***(ans.: 6.25)***

8. An express train reducing its velocity to *40* km./h. , has to apply the brakes for *50* sec. . If the retardation produced is *0.5* m./sec.^{2} , find its initial velocity in km./h. . ***(ans.: 130)***

9. At the instant from which time is measured a particle is passing through *O* and traveling towards *A* , along the straight line *OA*. It is s m. from *O* after *t* sec. where $s = t (t – 2)^2$:

i) when is it again at *O* ?

ii) when and where is it momentarily at rest ?

iii) what is the particle's greatest displacement from *O* , and how far does it moves , during the first *2* sec. ?

iv) what is the average velocity during the *3*rd sec. ?

v) at the end of the *1*st sec. where is the particle, which way is it going , and is its speed increasing or decreasing ?

vi) repeat (v) for the instant when $t = -1$.

(ans.:i)2;ii)0,32/27;iii)64/27;iv)3;v)OA;inceasing; vi)AO;decreasing)

10. A particle moves in a straight line so that after t sec. it is *s* m. , from a fixed point *O* on the line , where $s = t^4 + 3t^2$. Find :

i) The acceleration when $t = 1$, $t = 2$, and $t = 3$.

ii) The average acceleration between $t = 1$ and $t = 3$.
(ans.: i)18, 54,114; ii)58)

11. A particle moves along the x-axis in such away that its distance *x* cm. from the origin after *t* sec. is given by the formula $x = 27t – 2t^2$ what are its velocity and acceleration after *6.75* sec. ? How long does it take for the velocity to be reduced from *15* cm./sec. to *9* cm./sec., and how far does the particle travel mean while ? ***(ans.: 0,-4,1.5 ;18)***

12. A point moves along a straight line OX so that its distance x cm. from the point O at time t sec. is given by the formula $x = t^3 - 6t^2 + 9t$. Find :

i) at what times and in what positions the point will have zero velocity .

ii) its acceleration at these instants .

iii) its velocity when its acceleration is zero .

(ans.: i)1,3;4,0; ii)-6,6; iii)-3)

13. A particle moves in a straight line so that its distance x cm. from a fixed point O on the line is given by $x = 9t^2 - 2t^3$ where t is the time in seconds measured from O . Find the speed of the particle when $t = 3$. Also find the distance from O of the particle when $t = 4$, and show that it is then moving towards O . ***(ans.: 0, 16)***

14. Find the limits for the following functions by using L'Hopital's rule :

$1)\ \lim_{x\to\infty}\frac{5x^2-3x}{7x^2+1}$ $\quad 2)\ \lim_{t\to 0}\frac{\sin t^2}{t}$

$3)\ \lim_{x\to\frac{\pi}{2}}\frac{2x-\pi}{\cos x}$ $\quad 4)\ \lim_{t\to 0}\frac{\cos t-1}{t^2}$

$5)\ \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{1+\cos 2x}$ $\quad 6)\ \lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}}$

$7)\ \lim_{x\to 1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$ $\quad 8)\ \lim_{x\to 0}\frac{x(\cos x-1)}{\sin x-x}$

$9)\ \lim_{x\to 0} x.\csc^2\sqrt{2x}$ $\quad 10)\ \lim_{x\to 0}\frac{\sin x^2}{x.\sin x}$

$(ans.:1)\frac{5}{7};2)0;3)-2;4)-\frac{1}{2};5)\frac{1}{4};6)\sqrt{2};7)-1;8)3;9)\frac{1}{2};10)1)$

15. Find any local maximum and local minimum values , then sketch each curve by using first derivative :

1) $f(x) = x^3 - 4x^2 + 4x + 5$ $\quad (ans.: max.(0.7, 6.2); min.(2,5))$

2) $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ $\quad (ans.: min.(0,-1))$

3) $f(x) = x^5 - 5x - 6$ $\quad (ans.: max.(-1,-2); min.(1,-10))$

4) $f(x) = x^{\frac{4}{3}} - x^{\frac{1}{3}}$ $\quad (ans.: min.(0.25,-0.47))$

16. Find the interval of *x*-values on which the curve is concave up and concave down , then sketch the curve :

1) $f(x) = \dfrac{x^3}{3} + x^2 - 3x$ $\quad (ans.: up(-1,\infty); down(-\infty,-1))$

2) $f(x) = x^2 - 5x + 6$ $\quad (ans.: up(-\infty,\infty))$

3) $f(x) = x^3 - 2x^2 + 1$ $\quad (ans.: up(\frac{2}{3},\infty); down(-\infty,\frac{2}{3}))$

4) $f(x) = x^4 - 2x^2$ $\quad (ans.: up(-\infty,-\frac{1}{\sqrt{3}}),(\frac{1}{\sqrt{3}},\infty); down(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}))$

17. Sketch the following curve by using second derivative :

1) $y = \dfrac{x}{1 + x^2}$ $\quad (ans.: max.(1,0.5); min.(-1,-0.5))$

2) $y = -x(x-7)^2$ $\quad (ans.: max.(7,0); min.(2.3,-50.8))$

3) $y = (x+2)^2(x-3)$ $\quad (ans.: max.(-2,0); min.(1.3,-18.5))$

4) $y = x^2(5-x)$ $\quad (ans.: max.(3.3,18.5); min.(0,0))$

18. What is the smallest perimeter possible for a rectangle of area *16* in.^{2} ? *(ans.: 16)*

19. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the parabola $y = 12 - x^2$ **.** *(ans.:32)*

20) A rectangular plot is to be bounded on one side by a straight river and enclosed on the other three sides by a fence . With *800* m. of fence at your disposal . What is the largest area you can enclose ? *(ans.:80000)*

21) Show that the rectangle that has maximum area for a given perimeter is a square .

22) A wire of length *L* is available for making a circle and a square . How should the wire be divided between the two shapes to maximize the sum of the enclosed areas?

(ans.: all bent into a circle)

23) A closed container is made from a right circular cylinder of radius *r* and height *h* with a hemispherical dome on top . Find the relationship between *r* and *h* that maximizes the volume for a given surface area *s* . $(ans.: r = h = \sqrt{\frac{s}{5\pi}})$

24) An open rectangular box is to be made from a piece of cardboard *8* in. wide and *15* in. long by cutting a square from each corner and bending up the sides Find the dimensions of the box of largest volume . ***(ans.: height=5/3; width=14/3; length=35/3)***