MATHEMATICS 1

First Semester

For 1st class students

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References:

1- Thomas & Finney "Calculus and Analytic Geometry" (2005) , 11th edition , Addison Wesley.

2-Howard Anton, Irl Bivens & Stephen Davis "Calculus"(2009),9th edition ,John Wiley & Sons, INC.

Chapter One
Functions

Functions : ***Function*** **is any rule that assigns to each element in one set some element from another set :**

$$y = f(x)$$

The inputs make up the ***domain of the function*** **, and the outputs make up** ***the function's range.***

The variable x **is called** ***independent variable of the function*** **, and the variable** y **whose value depends on** x **is called** ***the dependent variable of the function*** **.**

We must keep two restrictions in mind when we define functions :

1. **We never divide by zero .**
2. **We will deal with real – valued functions only.**

Intervals :

- **The** ***open interval*** **is the set of all real numbers that be strictly between two fixed numbers** a **and** b **:**

$$(a,b) \equiv a < x < b$$

- **The** ***closed interval*** **is the set of all real numbers that contain both endpoints :**

$$[a,b] \equiv a \le x \le b$$

- ***Half open interval*** **is the set of all real numbers that contain one endpoint but not both :**

$$[a,b) \equiv a \le x < b$$

$$(a,b] \equiv a < x \le b$$

Composition of functions : **suppose that the outputs of a function** f **can be used as inputs of a function** g **. We can then hook** f **and** g **together to form a new function whose inputs are the inputs of** f **and whose outputs are the numbers :**

$$(g_o f)(x) = g(f(x))$$

EX-1- **Find the domain and range of each function :**

$$a)\quad y=\sqrt{x+4}\quad ,\quad b)\quad y=\frac{1}{x-2}$$

$$c)\quad y=\sqrt{9-x^2}\quad ,\quad d)\quad y=\sqrt{2-\sqrt{x}}$$

Sol. - $a)\quad x+4\geq 0\Rightarrow\quad x\geq -4\Rightarrow\quad D_x:\forall x\geq -4\quad ,\quad R_y:\forall y\geq 0$

$$b)\quad x-2\neq 0\Rightarrow\qquad x\neq 2\Rightarrow\qquad D_x:\forall x\neq 2$$

$$y=\frac{1}{x-2}\Rightarrow\quad x=\frac{1}{y}+2\Rightarrow\quad R_y:\forall y\neq 0$$

$$c)\qquad 9-x^2\geq 0\Rightarrow\qquad -3\leq x\leq 3\Rightarrow\quad D_x:-3\leq x\leq 3$$

$$y=\sqrt{9-x^2}\Rightarrow\qquad x=\mp\sqrt{9-y^2}$$

$$since\quad 9-y^2\geq 0\Rightarrow\quad -3\leq y\leq 3$$

$$since\quad y\geq 0\qquad\qquad \Rightarrow\qquad R_y:0\leq y\leq 3$$

$$d)\qquad 2-\sqrt{x}\geq 0\Rightarrow\qquad 0\leq x\leq 4\Rightarrow\quad D_x:0\leq x\leq 4$$

$$\left.\begin{array}{l} if\quad x=0\Rightarrow\quad y=\sqrt{2} \\ if\quad x=4\Rightarrow\quad y=0\end{array}\right\}\Rightarrow\qquad R_y:0\leq y\leq\sqrt{2}$$

EX-2- **Let** $f(x)=\dfrac{x}{x-1}$ *and* $g(x)=1+\dfrac{1}{x}$.

Find $(g_of)(x)$ **and** $(f_og)(x)$.

Sol.-

$$(g_o f)(x)=g(f(x))=g\left(\frac{x}{x-1}\right)=1+\frac{1}{\frac{x}{x-1}}=\frac{2x-1}{x}$$

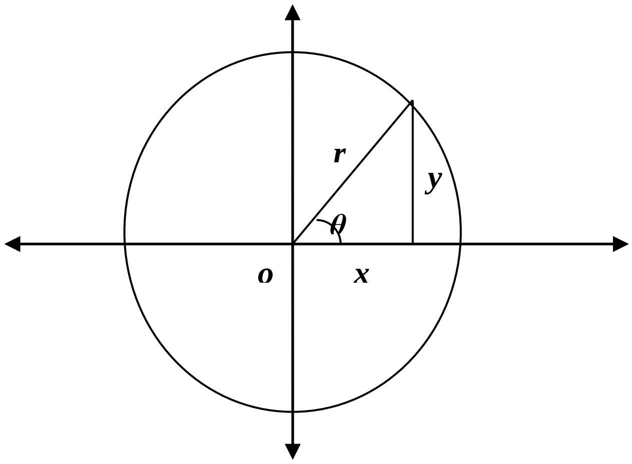
$$(f_o g)(x)=f(g(x))=f\left(1+\frac{1}{x}\right)=\frac{1+\frac{1}{x}}{1+\frac{1}{x}-1}=x+1$$

EX-3- **Let** $(g_o f)(x)=x$ *and* $f(x)=\dfrac{1}{x}$. **Find** $g(x)$.

Sol.- $(g_o f)(x)=g\left(\dfrac{1}{x}\right)=x\Rightarrow\quad g(x)=\dfrac{1}{x}$

<u>Trigonometric functions</u> : When an angle of measure θ is placed in standard position at the center of a circle of radius r , the trigonometric functions of θ are defined by the equations :

$$Sin\theta = \frac{y}{r} = \frac{1}{csc\,\theta} \quad , \quad Cos\theta = \frac{x}{r} = \frac{1}{sec\,\theta} \quad , \quad tan\,\theta = \frac{y}{x} = \frac{1}{Cot\theta} = \frac{Sin\theta}{Cos\theta}$$



The following are some properties of these functions :

1) $Sin^2\theta + Cos^2\theta = 1$

2) $1 + tan^2\,\theta = sec^2\,\theta$ and $1 + Cot^2\theta = csc^2\,\theta$

3) $Sin(\theta \mp \beta) = Sin\theta.Cos\beta \mp Cos\theta.Sin\beta$

4) $Cos(\theta \mp \beta) = Cos\theta.Cos\beta \pm Sin\theta.Sin\beta$

5) $tan(\theta \mp \beta) = \dfrac{tan\,\theta \mp tan\,\beta}{1 \pm tan\,\theta.tan\,\beta}$

6) $Sin2\theta = 2Sin\theta.Cos\theta$ and $Cos2\theta = Cos^2\theta - Sin^2\theta$

7) $Cos^2\theta = \dfrac{1 + Cos2\theta}{2}$ and $Sin^2\theta = \dfrac{1 - Cos2\theta}{2}$

8) $Sin(\theta \mp \frac{\pi}{2}) = \mp Cos\theta$ and $Cos(\theta \mp \frac{\pi}{2}) = \pm Sin\theta$

9) $Sin(-\theta) = -Sin\theta$ and $Cos(-\theta) = Cos\theta$ and $tan(-\theta) = -tan\,\theta$

10) $Sin\theta.Sin\beta = \frac{1}{2}[Cos(\theta - \beta) - Cos(\theta + \beta)]$

$Cos\theta.Cos\beta = \frac{1}{2}[Cos(\theta - \beta) + Cos(\theta + \beta)]$

$Sin\theta.Cos\beta = \frac{1}{2}[Sin(\theta - \beta) + Sin(\theta + \beta)]$

$$11)\quad Sin\theta + Sin\beta = 2Sin\frac{\theta+\beta}{2}.Cos\frac{\theta-\beta}{2}$$

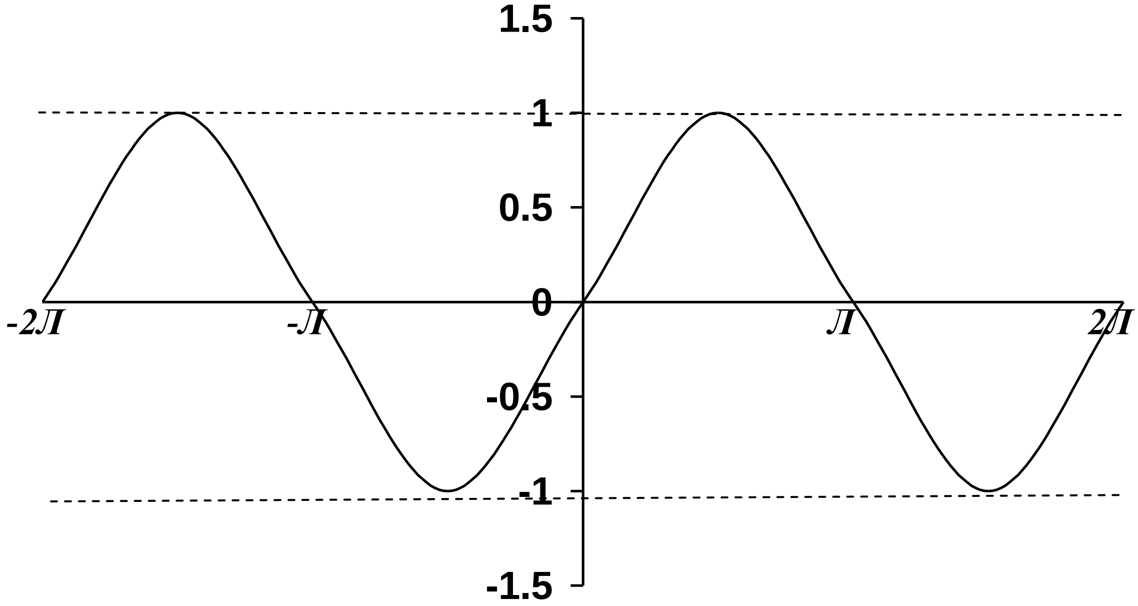
$$Sin\theta - Sin\beta = 2Cos\frac{\theta+\beta}{2}.Sin\frac{\theta-\beta}{2}$$

$$12)\quad Cos\theta + Cos\beta = 2Cos\frac{\theta+\beta}{2}.Cos\frac{\theta-\beta}{2}$$

$$Cos\theta - Cos\beta = -2Sin\frac{\theta+\beta}{2}.Sin\frac{\theta-\beta}{2}$$

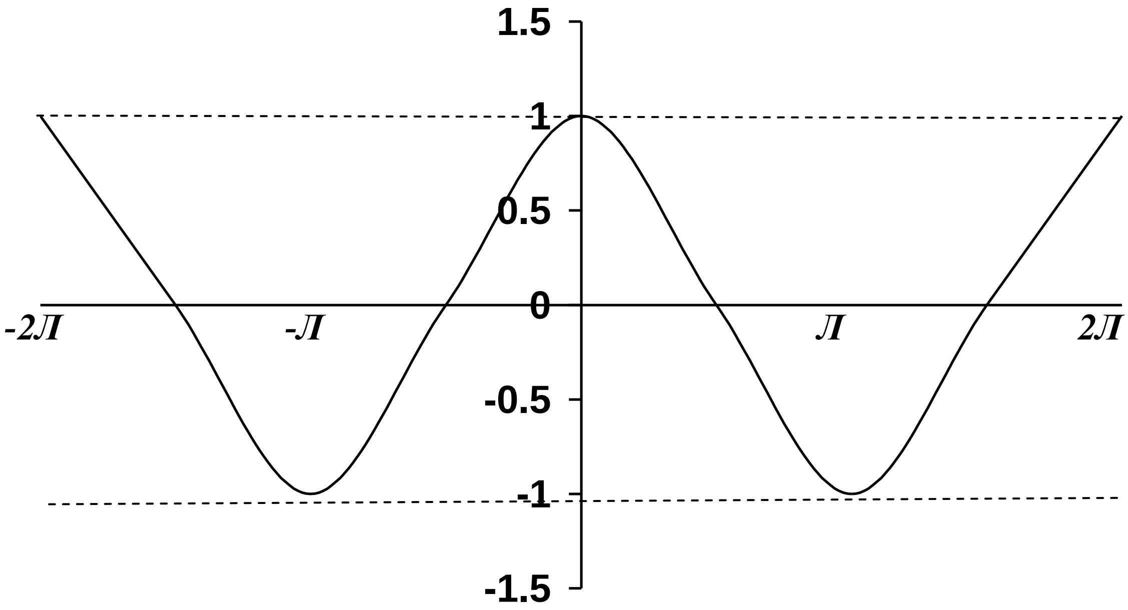
θ	*0*	*П / 6*	*П / 4*	*П / 3*	*П / 2*	*П*
Sinθ	*0*	*1/2*	*1/√2*	*√3/2*	*1*	*0*
Cosθ	*1*	*√3/2*	*1/√2*	*1/2*	*0*	*-1*
tanθ	*0*	*1/√3*	*1*	*√3*	∞	*0*

Graphs of the trigonometric functions are :

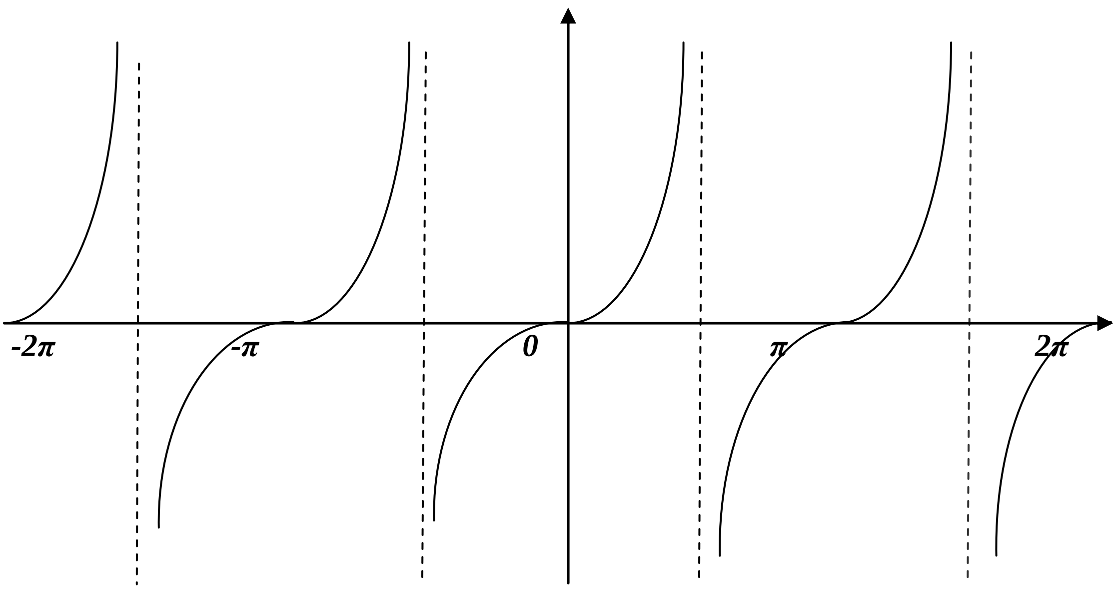


$$y = Sinx \quad D_x : \forall x$$

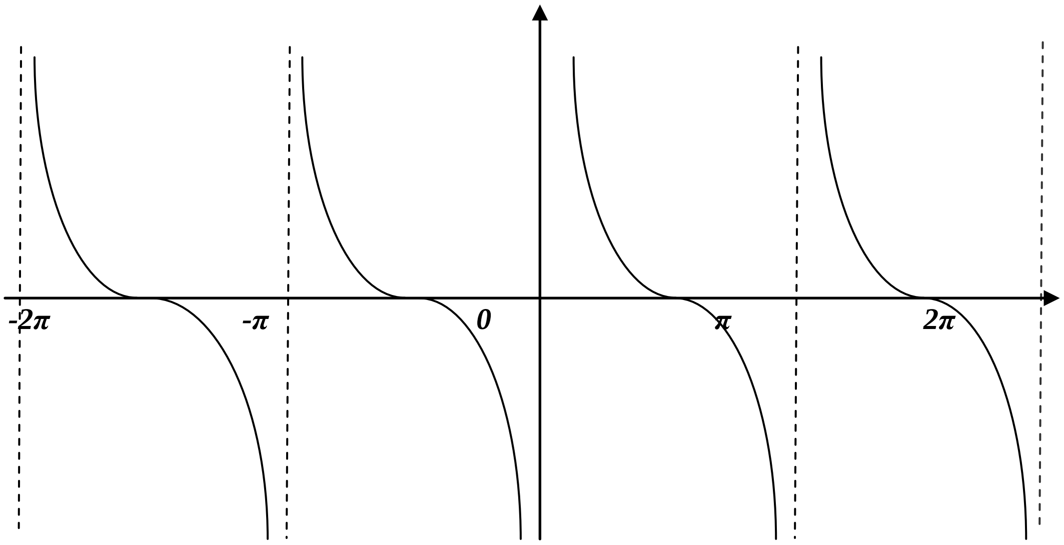
$$R_y : -1 \le y \le 1$$



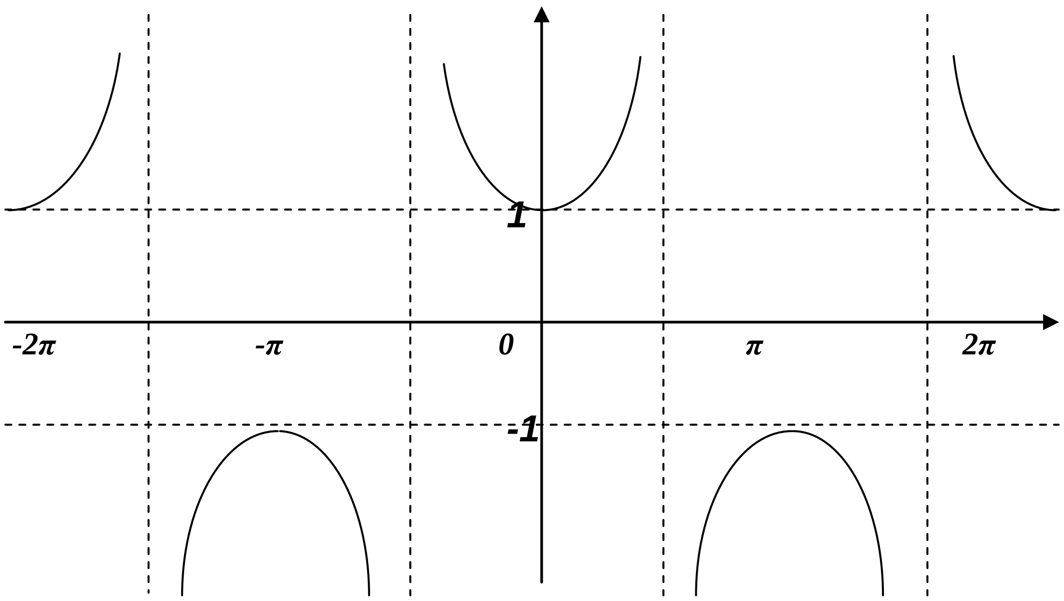
$$y = Cosx \quad D_x : \forall x$$
$$R_y : -1 \leq y \leq 1$$



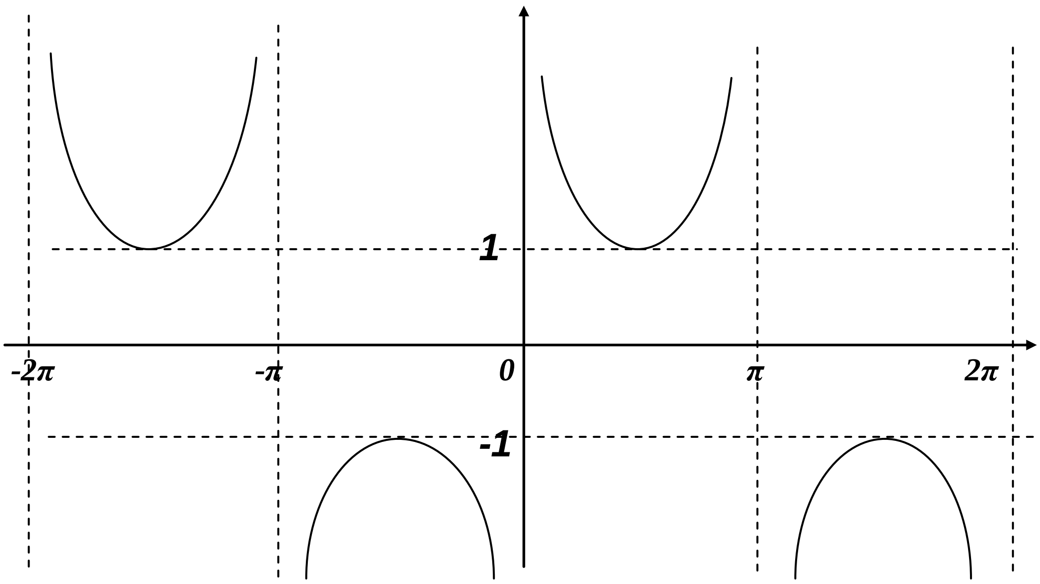
$$y = \tan x \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$
$$R_y : \forall y$$



$$y = Cotx \quad D_x : \forall x \neq n\pi$$
$$R_y : \forall y$$



$$y = Secx \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$
$$R_y : \forall y \geq 1 \ \ or \ \ y \leq -1$$



$$y = \csc x \quad D_x : \forall x \neq n\pi$$
$$R_y : \forall y \geq 1 \ \ or \ \ y \leq -1$$

$$Where \ \ n = 0, \mp 1, \mp 2, \mp 3, \ldots\ldots$$

<u>*EX-2*</u> - Solve the following equations , for values of θ from 0^o to 360^o inclusive .

a) $\tan\theta = 2\sin\theta$ **b)** $1 + \cos\theta = 2\sin^2\theta$

<u>*Sol.*</u>-

$$a) \quad \tan\theta = 2\sin\theta \Rightarrow \frac{\sin\theta}{\cos\theta} = 2\sin\theta$$
$$\Rightarrow \sin\theta(1 - 2\cos\theta) = 0$$
$$either \ \ \sin\theta = 0 \Rightarrow \theta = 0^o, 180^o, 360^o$$
$$or \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$

Therefore the required values of θ are $0^o, 60^o, 180^o, 300^o, 360^o$.

$$b) \quad 1 + \cos\theta = 2.\sin^2\theta \Rightarrow 1 + \cos\theta = 2(1 - \cos^2\theta)$$
$$\Rightarrow (2\cos\theta - 1)(\cos\theta + 1) = 0$$
$$either \quad \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$$

$$or \quad \cos\theta = -1 \Rightarrow \theta = 180^o$$

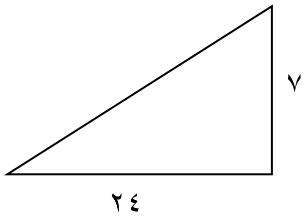
There the roots of the equation between 0^o and 360^o are $60^o, 180^o$ and 300^o .

EX-3- **If** ***tan θ = 7/24*, find without using tables the values of** ***Secθ*** **and** ***Sinθ*.**

Sol.-

$$\tan\theta = \frac{y}{x} = \frac{7}{24} \Rightarrow r = \sqrt{7^2 + 24^2} = 25$$

$$Sec\theta = \frac{r}{x} = \frac{25}{24} \quad and \quad Sin\theta = \frac{y}{r} = \frac{7}{25}$$



EX-4- **Prove the following identities :**

$$a)\quad Csc\theta + \tan\theta.Sec\theta = Csc\theta.Sec^2\theta$$

$$b)\quad Cos^4\theta - Sin^4\theta = Cos^2\theta - Sin^2\theta$$

$$c)\quad \frac{Sec\theta - Csc\theta}{\tan\theta - Cot\theta} = \frac{\tan\theta + Cot\theta}{Sec\theta + Csc\theta}$$

Sol.-

$$a)\quad L.H.S. = Csc\theta + \tan\theta.Sec\theta = \frac{1}{Sin\theta} + \frac{Sin\theta}{Cos\theta}.\frac{1}{Cos\theta}$$

$$= \frac{Cos^2\theta + Sin^2\theta}{Sin\theta.Cos^2\theta} = \frac{1}{Sin\theta}.\frac{1}{Cos^2\theta} = Csc\theta.Sec^2\theta = R.H.S.$$

$$b)\quad L.H.S. = Cos^4\theta - Sin^4\theta = (Cos^2\theta - Sin^2\theta).(Cos^2\theta + Sin^2\theta)$$

$$= Cos^2\theta - Sin^2\theta = R.H.S.$$

$$c)\quad L.H.S. = \frac{Sec\theta - Csc\theta}{\tan\theta - Cot\theta} = \frac{\frac{1}{Cos\theta} - \frac{1}{Sin\theta}}{\frac{Sin\theta}{Cos\theta} - \frac{Cos\theta}{Sin\theta}} = \frac{1}{Sin\theta + Cos\theta}$$

$$= \frac{Sin^2\theta + Cos^2\theta}{Sin\theta + Cos\theta}.\frac{\frac{1}{Sin\theta.Cos\theta}}{\frac{1}{Sin\theta.Cos\theta}} = \frac{\tan\theta + Cot\theta}{Sec\theta + Csc\theta} = R.H.S.$$

EX-5- **Simplify** $\frac{1}{\sqrt{x^2 - a^2}}$ *when* $x = a.Csc\theta$.

Sol.- $\frac{1}{\sqrt{x^2 - a^2}} = \frac{1}{\sqrt{a^2Csc^2\theta - a^2}} = \frac{1}{a\sqrt{Cot^2\theta}} = \frac{1}{a}\tan\theta$.

EX-6- **Eliminate** ***θ*** **from the equations :**

i) $x = a\,Sin\theta$ *and* $y = b\,tan\theta$

ii) $x = 2\,Sec\theta$ *and* $y = Cos2\theta$

Sol.-

$$i)\qquad x = a.Sin\theta \Rightarrow Sin\theta = \frac{x}{a} \Rightarrow Csc\theta = \frac{a}{x}$$

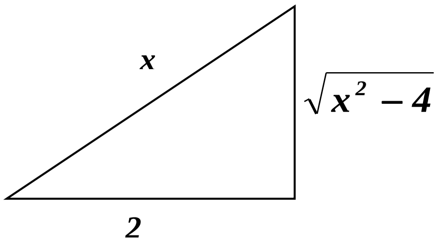
$$y = b\tan\theta \Rightarrow \tan\theta = \frac{y}{b} \Rightarrow Cot\theta = \frac{b}{y}$$

$$Since \quad Csc^2\theta = Cot^2\theta + 1 \Rightarrow \frac{a^2}{x^2} = \frac{b^2}{y^2} + 1$$

$$ii)\qquad x = 2Sec\theta \Rightarrow Cos\theta = \frac{2}{x}$$

$$y = Cos2\theta \Rightarrow y = Cos^2\theta - Sin^2\theta$$

$$y = \frac{4}{x^2} - \frac{x^2 - 4}{x^2} \Rightarrow x^2 y = 8 - x^2$$



EX-7- If $tan^2\theta - 2\ tan^2\beta = 1$, show that $2\ Cos^2\theta - Cos^2\beta = 0$.

Sol. –

$$tan^2\ \theta - 2\tan^2\ \beta = 1 \Rightarrow Sec^2\theta - 1 - 2(Sec^2\beta - 1) = 1$$

$$\Rightarrow Sec^2\theta - 2Sec^2\beta = 0 \Rightarrow \frac{1}{Cos^2\theta} - \frac{2}{Cos^2\beta} = 0$$

$$\Rightarrow 2Cos^2\theta - Cos^2\beta = 0 \qquad Q.E.D.$$

EX-8- If $a\ Sin\theta = p - b\ Cos\theta$ and $b\ Sin\theta = q + a\ Cos\theta$ .Show that :
$a^2 + b^2 = p^2 + q^2$

Sol.-

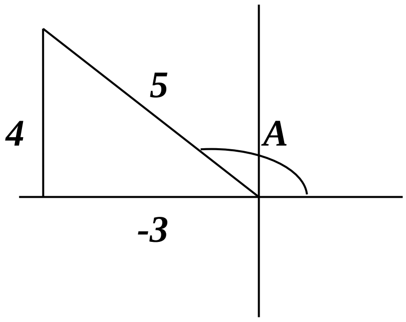
$$p = a.Sin\theta + b.Cos\theta \quad and \quad q = b.Sin\theta - a.Cos\theta$$

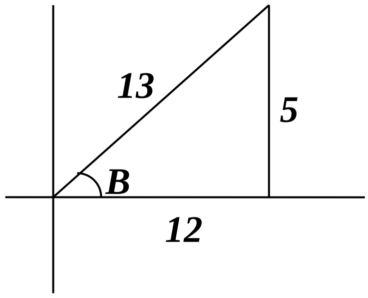
$$p^2 + q^2 = (aSin\theta + bCos\theta)^2 + (bSin\theta - aCos\theta)^2$$

$$= a^2(Sin^2\theta + Cos^2\theta) + b^2(Cos^2\theta + Sin^2\theta) = a^2 + b^2$$

EX-9- If $Sin\ A = 4/5$ and $Cos\ B = 12/13$,where A is obtuse and B is acute . Find , without tables , the values of :
a) $Sin\ (A - B)$, b) $tan\ (A - B)$, c) $tan\ (A + B)$.

<u>Sol.</u> -





$$a)\quad Sin(A-B)=SinA.CosB-CosA.SinB$$

$$=\frac{4}{5}.\frac{12}{13}+\frac{3}{5}.\frac{5}{13}=\frac{63}{65}$$

$$b)\qquad tan(A-B)=\frac{tan\,A-tan\,B}{1+tan\,A.tan\,B}$$

$$=\frac{-\frac{4}{3}-\frac{5}{12}}{1-\frac{4}{3}.\frac{5}{12}}=-\frac{63}{16}$$

$$c)\qquad tan(A+B)=\frac{tan\,A+tan\,B}{1-tan\,A.tan\,B}$$

$$=\frac{-\frac{4}{3}+\frac{5}{12}}{1+\frac{4}{3}.\frac{5}{12}}=-\frac{33}{56}$$

<u>*EX-10*</u> – Prove the following identities:

$$a)\quad Sin(A+B)+Sin(A-B)=2.SinA.CosB$$

$$b)\quad tan\,A+tan\,B=\frac{Sin(A+B)}{CosA.CosB}$$

$$c)\quad Sec(A+B)=\frac{SecA.SecB.CscA.CscB}{CscA.CscB-SecA.SecB}$$

$$d)\quad \frac{Sin2\theta+Cos2\theta+1}{Sin2\theta-Cos2\theta+1}=Cot\theta$$

Sol.-

$$a)\quad L.H.S. \quad = Sin(A+B)+Sin(A-B)$$

$$= SinA.CosB + CosA.SinB + SinA.CosB - CosA.SinB$$

$$= 2.SinA.CosB = R.H.S.$$

$$b)\quad R.H.S. \quad = \frac{Sin(A+B)}{CosA.CosB} = \frac{SinA.CosB + CosA.SinB}{CosA.CosB}$$

$$= tan\,A + tan\,B = L.H.S.$$

$$c)\quad R.H.S \quad = \frac{SecA.SecB.CscA.CscB}{CscA.CscB - SecA.SecB} = \frac{\frac{1}{CosA}.\frac{1}{CosB}.\frac{1}{SinA}.\frac{1}{SinB}}{\frac{1}{SinA}.\frac{1}{SinB} - \frac{1}{CosA}.\frac{1}{CosB}}$$

$$= \frac{1}{CosA.CosB - SinA.SinB} = \frac{1}{Cos(A+B)}$$

$$= Sec(A+B) = L.H.S.$$

$$d)\quad L.H.S. \quad = \frac{Sin2\theta + Cos2\theta + 1}{Sin2\theta - Cos2\theta + 1} = \frac{2Sin\theta.Cos\theta + (Cos^2\theta - Sin^2\theta) + 1}{2Sin\theta.Cos\theta - (Cos^2\theta - Sin^2\theta) + 1}$$

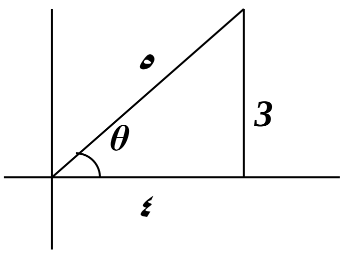
$$= \frac{2Sin\theta.Cos\theta + 2Cos^2\theta}{2Sin\theta.Cos\theta + 2Sin^2\theta} = \frac{Cos\theta}{Sin\theta} = Cot\theta = R.H.S.$$

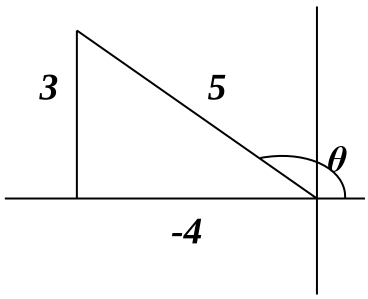
EX-11 – **Find , without using tables , the values of** ***Sin 2θ*** **and** ***Cos 2θ***, **when:**

a) $Sin\theta = 3/5$, **b)** $Cos\,\theta = 12/13$, **c)** $Sin\,\theta = -\sqrt{3}/2$.

Sol. –

a)

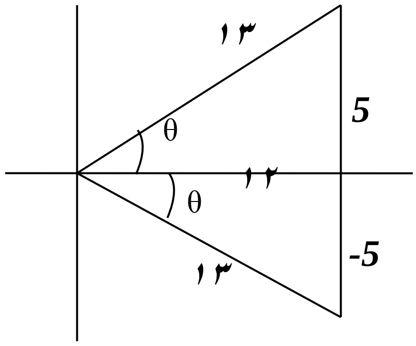




$$Sin2\theta = 2.Sin\theta.Cos\theta = 2.\frac{3}{5}.(\mp\frac{4}{5}) = \mp\frac{24}{25}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\mp\frac{4}{5})^2 - (\frac{3}{5})^2 = \frac{7}{25}$$

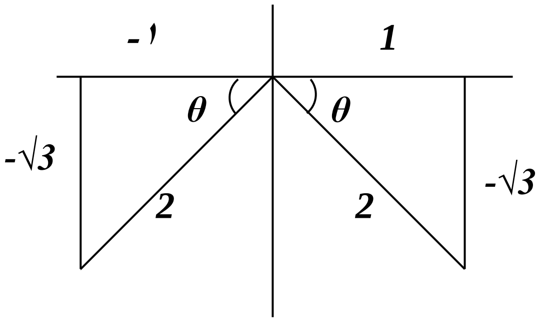
b)



$$Sin2\theta = 2.Sin\theta.Cos\theta = 2(\mp\frac{5}{13}).(\frac{12}{13}) = \mp\frac{120}{169}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\frac{12}{13})^2 - (\mp\frac{5}{13})^2 = \frac{119}{169}$$

c)



$$Sin2\theta = 2Sin\theta.Cos\theta = 2(-\frac{\sqrt{3}}{2}).(\mp\frac{1}{2}) = \pm\frac{\sqrt{3}}{2}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\mp\frac{1}{2})^2 - (-\frac{\sqrt{3}}{2})^2 = -\frac{1}{2}$$

<u>*EX-12-*</u> Solve the following equations for values of *θ* from *0^o* to *360^o* inclusive:

a) $Cos\ 2\theta + Cos\ \theta + 1 = 0$, **b)** $4\ tan\ \theta\ .\ tan\ 2\theta = 1$

Sol.-

a) $Cos2\theta + Cos\theta + 1 = 0 \Rightarrow 2Cos^2\theta - 1 + Cos\theta + 1 = 0$

$$\Rightarrow Cos(2.Cos\theta + 1) = 0$$

either $Cos\theta = 0 \Rightarrow \theta = 90^o, 270^o$

or $Cos\theta = -\frac{1}{2} \Rightarrow \theta = 120^o, 240^o$

$$\theta = \{90^o, 120^o, 240^o, 270^o\}$$

b) $4.\tan\theta.\tan 2\theta = 1 \Rightarrow 4.\tan\theta.\frac{2\tan\theta}{1-\tan^2\theta} = 1$

$$\Rightarrow 9\tan^2\theta = 1$$

either $\tan\theta = \frac{1}{3} \Rightarrow \theta = 18.4^o, 198.4^o$

or $\tan\theta = -\frac{1}{3} \Rightarrow \theta = 161.6^o, 341.6^o$

$$\theta = \{18.4^o, 161.6^o, 198.4^o, 341.6^o\}$$

Problems (1)

1. Let $y=\dfrac{x^2+2}{x^2-1}$ **, express** x **in terms of** y **and find the values of** y **for which** x **is real .** **(ans.:** $x=\mp\sqrt{\dfrac{y+2}{y-1}};\ y\le -2\ \ or\ \ y>1)$

2. Find the domain and range of each function :

$$a)\ y=\frac{1}{1+x^2}\quad ,\quad b)\ y=\frac{1}{1+\sqrt{x}}\quad ,\quad c)\ y=\frac{1}{\sqrt{3-x}}$$

$(ans.:a)\forall x, 0<y\le 1;\ b)x\ge 0, y>0;\ c)x<3, y>0)$

3. Find the points of intersection of $x^2 = 4y$ **and** $y = 4x$ **.** ***(ans.:(0,0),(16,64))***

4. Find the coordinates of the points at which the curves cut the axes :

$$a)\ y=x^3-9x^2\quad ,\quad b)\ y=(x^2-1)(x^2-9)\quad ,\quad c)\ y=(x+1)(x-5)^2$$

(ans.:a)(0,0);(0,0),(9,0);b)(0,9);(1,0),(-1,0),(3,0),(-3,0);c)(0,25);(-1,0),(5,0))

5. Let $f(x) = ax + b$ **and** $g(x) = cx + d$ **. What condition must be satisfied by the constants** a **,** b **,** c **and** d **to make** $f(g(x))$ **and** $g(f(x))$ **identical ?**
(ans.:ad+b=bc+d)

6. If $f(x) = 1/x$ **and** $g(x)=1/\sqrt{x}$ **, what are the domain of** f **,** g **,** $f+g$ **,** $f-g$ **,** $f.g$ **,** f/g **,** g/f **,** $f_o g$ **and** $g_o f$ **? What is the domain of** $h(x) = g(x+4)$ **?**

$(ans.: \forall x\ne 0, \forall x>0, \forall x>0, \forall x>0, \forall x>0, \forall x>0, \forall x\ge 0, \forall x\ge 0, \forall x\ge 0; \forall x>-4)$

7. Solve the following equations for values of θ **from** -180^o **to** 180^o **inclusive:**

i) $\tan^2\theta + \tan\theta = 0$ ***ii)*** $Cot\ \theta = 5\ Cos\ \theta$

iii) $3\ Cos\ \theta + 2\ Sec\ \theta + 7 = 0$ ***iv)*** $Cos^2\theta + Sin\ \theta + 1 = 0$

(ans.:i)-180,-45,0,135,180; ii)-90,11.5,90,168.5; iii)-109.5,109.5; iv)-90)

8. Solve the following equations for values of θ **from** 0^o **to** 360^o **inclusive:**

i) $3\ Cos\ 2\theta - Sin\ \theta + 2 = 0$ ***ii)*** $3\tan\theta = \tan 2\theta$

iii) $Sin\ 2\theta.\ Cos\ \theta + Sin^2\theta = 1$ ***iv)*** $3\ Cot\ 2\theta + Cot\ \theta = 1$

(ans.:i)56.4,123.6,270; ii)0,30,150,180,210,330,360; iii)30,90,150,270; iv)45,121,225,301)

9. If $Sin\ \theta = 3/5$ **, find without using tables the values of :**

i) $Cos\ \theta$ ***ii)*** $\tan\theta$ ***(ans.: i) 4/5 ; ii) 3/4)***

10. Find, without using tables, the values of $Cos\ x$ **and** $Sin\ x$**,when** $Cos\ 2x$ **is :**

a) 1/8 , b) 7/25 , c) -119/169

$$(ans.: a)\mp\frac{3}{4},\mp\frac{\sqrt{7}}{4}; b)\mp\frac{4}{5},\mp\frac{3}{5}; c)\mp\frac{5}{13},\mp\frac{12}{13})$$

11. If *Sin A = 3/5* and Sin *B = 5/13* , where *A* and *B* are acute angles , find without using tables , the values of :
a) Sin(A+B) , b) Cos(A+B) , c) Cot(A+B) (ans.: 56/65; 33/65; 33/56)

12. If tan *A = -1/7* and *tan B = 3/4* , where *A* is obtuse and *B* is acute , find without using tables the value of *A – B* . ***(ans.: 135)***

13.Prove the following identities :

1) $\mathrm{Sec}^2\theta+\mathrm{Csc}^2\theta=\mathrm{Sec}^2\theta.\mathrm{Csc}^2\theta$

2) $\mathrm{Sin}^2\theta(1+\mathrm{Sec}^2\theta)=\mathrm{Sec}^2\theta-\mathrm{Cos}^2\theta$

3) $\dfrac{1+\mathrm{Sin}\theta}{1-\mathrm{Sin}\theta}=(\mathrm{Sec}\theta+\tan\theta)^2$

4) $\mathrm{Sec}\theta-\mathrm{Sin}\theta=\dfrac{\tan^2\theta+\mathrm{Cos}^2\theta}{\mathrm{Sec}\theta+\mathrm{Sin}\theta}$

5) $\dfrac{\mathrm{Cos}(A-B)-\mathrm{Cos}(A+B)}{\mathrm{Sin}(A+B)+\mathrm{Sin}(A-B)}=\tan B$

6) $\mathrm{Cos}B-\mathrm{Cos}A.\mathrm{Cos}(A-B)=\mathrm{Sin}A.\mathrm{Sin}(A-B)$

7) $\tan(A+B+C)=\dfrac{\tan A+\tan B+\tan C-\tan A.\tan B.\tan C}{1-\tan B.\tan C-\tan C.\tan A-\tan A.\tan B}$

If A,B,C are angles of a triangle, show that :

$\tan A+\tan B+\tan C=\tan A.\tan B.\tan C$

8) $\tan x=\sqrt{\dfrac{1-\mathrm{Cos}2x}{1+\mathrm{Cos}2x}}$

9) $\dfrac{\mathrm{Sin}4A+\mathrm{Sin}2A}{\mathrm{Cos}4A+\mathrm{Cos}2A+1}=\tan 2A$

10) $\mathrm{Sin}^4\theta+\mathrm{Cos}^4\theta=\dfrac{1}{4}(\mathrm{Cos}4\theta+3)$

11) $4\mathrm{Sin}^3A.\mathrm{Cos}3A+4\mathrm{Cos}^3A.\mathrm{Sin}3A=3\mathrm{Sin}4A$

12) $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

14. If $u=\dfrac{1+Sin\theta}{Cos\theta}$**, prove that** $\dfrac{1}{u}=\dfrac{1-Sin\theta}{Cos\theta}$ **and deduce formula for** $Sin\theta$ **,** $Cos\theta$ **,** $tan\theta$ **in terms of *u*.** ***(ans.:***$(u^2-1)/(u^2+1)$; $2u/(u^2+1)$; $(u^2-1)/(u^2+1)$***)***

15. If $Sin(x+\alpha)=2Cos(x-\alpha);$ prove that : $tan\,x=\dfrac{2-tan\alpha}{1-2tan\alpha}$.

16. If $Sin(x-\alpha)=Cos(x+\alpha);$ prove that : $tan\,x=1$.

17. If $x=Cos\theta+Cos2\theta$ $\quad and \quad$ $y=Sin\theta+Sin2\theta$. Show that :

i) $x^2-y^2=Cos2\theta+2Cos3\theta+Cos4\theta$

ii) $2xy=Sin2\theta+2Sin3\theta+Sin4\theta$

18. If $Cos2A.Cos2B=Cos2\theta$, prove that :
$Sin^2A.Cos^2B+Cos^2A.Sin^2B=Sin^2\theta$

19. If $S=Sin\ \theta$ and $C=Cos\ \theta$, simplify :

i) $\dfrac{S.C}{\sqrt{1-S^2}}$, ii) $\dfrac{S.\sqrt{1-S^2}}{C.\sqrt{1-C^2}}$, iii) $\dfrac{C}{S}+\dfrac{S}{C}$

(ans.:i) Sinθ; ii)1; iii) Secθ.Cscθ)

Chapter Two
Limits and continuity

Limits : **The limit of $F(t)$ as t approaches C is the number L if :**
Given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about C such that for all t, $0 < |t - C| < \delta$ implies $|F(t) - L| < \varepsilon$ and we can write it as :

$$\lim_{t \to C} F(t) = L$$

The limit of a function $F(t)$ as $t \to C$ never depend on what happens when $t = C$.

Right hand limit : $\lim_{t \to C^+} F(t) = L$

The limit of the function $F(t)$ as $t \to C$ from the right equals L if :
Given any $\varepsilon > 0$ (radius about L) there exists a $\delta > 0$ (radius to the right of C) such that for all t :

$$C < t < C + \delta \Rightarrow \quad |F(t) - L| < \varepsilon$$

Left hand limit : $\lim_{t \to C^-} F(t) = L$

The limit of the function $F(t)$ as $t \to C$ from the left equal L if :
Given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all t :

$$C - \delta < t < C \Rightarrow \quad |F(t) - L| < \varepsilon$$

Note that – **A function $F(t)$ has a limit at point C if and only if the right hand and the left hand limits at C exist and equal . In symbols :**

$$\lim_{t \to C} F(t) = L \quad \Leftrightarrow \quad \lim_{t \to C^+} F(t) = L \quad \text{and} \quad \lim_{t \to C^-} F(t) = L$$

The limit combinations theorems :

$$1) \quad \lim [F_1(t) \mp F_2(t)] = \lim F_1(t) \mp \lim F_2(t)$$

$$2) \quad \lim [F_1(t) * F_2(t)] = \lim F_1(t) * \lim F_2(t)$$

$$3) \quad \lim \frac{F_1(t)}{F_2(t)} = \frac{\lim F_1(t)}{\lim F_2(t)} \quad \text{where} \quad \lim F_2(t) \neq 0$$

$$4) \quad \lim [k * F_1(t)] = k * \lim F_1(t) \quad \forall k$$

$$5) \quad \lim_{\theta \to 0} \frac{Sin\theta}{\theta} = 1$$

provided that θ is measured in radius

The limits (in $1 - 4$) are all to be taken as $t \to C$ and $F_1(t)$ and $F_2(t)$ are to be real functions .

Thm. -1 : The sandwich theorem **:** **Suppose that** $f(t) \le g(t) \le h(t)$ **for all** $t \ne C$ **in some interval about** C **and that** $f(t)$ **and** $h(t)$ **approaches the same limit** L **as** $t \to C$ **, then :**

$$\lim_{t \to C} g(t) = L$$

Infinity as a limit **:**

1.The limit of the function $f(x)$ **as** x **approaches infinity is the number** L**:** $\lim_{x \to \infty} f(x) = L$ **. If , given any** $\varepsilon > 0$ **there exists a number** M **such that for all** x **:** $M < x \quad \Rightarrow \quad |f(x) - L| < \varepsilon$ **.**

2. The limit of $f(x)$ **as** x **approaches negative infinity is the number** L **:** $\lim_{x \to -\infty} f(x) = L$ **. If , given any** $\varepsilon > 0$ **there exists a number** N **such that for all** x **:** $x < N \quad \Rightarrow \quad |f(x) - L| < \varepsilon$ **.**

The following facts are some times abbreviated by saying :

a) **As** x **approaches** 0 **from the right ,** $1/x$ **tends to** ∞ **.**
b) **As** x **approaches** 0 **from the left ,** $1/x$ **tends to** $-\infty$ **.**
c) **As** x **tends to** ∞ **,** $1/x$ **approaches** 0 **.**
d) **As** x **tends to** $-\infty$ **,** $1/x$ **approaches** 0 **.**

Continuity **:**

Continuity at an interior point **: A function** $y = f(x)$ **is continuous at an interior point** C **of its domain if :** $\lim_{x \to C} f(x) = f(C)$ **.**

Continuity at an endpoint **: A function** $y = f(x)$ **is continuous at a left endpoint** a **of its domain if :** $\lim_{x \to a^+} f(x) = f(a)$ **.**

A function $y = f(x)$ **is continuous at a right endpoint** b **of its domain if:** $\lim_{t \to b^-} f(x) = f(b)$ **.**

Continuous function **: A function is continuous if it is continuous at each point of its domain .**

Discontinuity at a point **: If a function** f **is not continuous at a point** C **, we say that** f **is discontinuous at** C **, and call** C **a point of discontinuity of** f **.**

The continuity test **: The function $y = f(x)$ is continuous at $x = C$ if and only if all three of the following statements are true :**

1) $f(C)$ **exist (C is in the domain of f) .**
2) $\lim\limits_{x\to C} f(x)$ **exists (f has a limit as $x \to C$) .**
3) $\lim\limits_{x\to C} f(x) = f(C)$ **(the limit equals the function value) .**

Thm.-2 : The limit combination theorem for continuous function :

If the function f and g are continuous at $x = C$, then all of the following combinations are continuous at $x = C$:

$1)\ f \mp g \quad 2)\ f.g \quad 3)\ k.g\ \ \forall k \quad 4)\ g_o f\ ,\ f_o g \quad 5)\ f / g$

provided $g(C) \neq 0$

Thm.-3 **: A function is continuous at every point at which it has a derivative . That is , if $y = f(x)$ has a derivative $f'(C)$ at $x = C$, then f is continuous at $x = C$.**

EX-4 **– Find :**

$1)\ \lim\limits_{x\to 0}\dfrac{5x^3+8x^2}{3x^4-16x^2}$, $2)\ \lim\limits_{x\to a}\dfrac{x^3-a^3}{x^4-a^4}$

$3)\ \lim\limits_{x\to 0}\dfrac{Sin5x}{Sin3x}$, $4)\ \lim\limits_{y\to 0}\dfrac{tan2y}{3y}$

$5)\ \lim\limits_{x\to 0}\dfrac{Sin2x}{2x^2+x}$, $6)\ \lim\limits_{x\to\infty}\left(1+Cos\dfrac{1}{x}\right)$

$7)\ \lim\limits_{x\to\infty}\dfrac{3x^3+5x^2-7}{10x^3-11x+5}$, $8)\ \lim\limits_{y\to\infty}\dfrac{3y+7}{y^2-2}$

$9)\ \lim\limits_{x\to\infty}\dfrac{x^3-1}{2x^2-7x+5}$, $10)\ \lim\limits_{x\to -1^-}\dfrac{1}{x+1}$

$11)\ \lim\limits_{x\to 0}Cos\left(1-\dfrac{Sinx}{x}\right)$, $12)\ \lim\limits_{x\to 0}Sin\left(\dfrac{\pi}{2}Cos(tan\,x)\right)$

S0l.-

$$1)\ \lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}=\lim_{x\to 0}\frac{5x+8}{3x^2-16}=\frac{0+8}{0-16}=-\frac{1}{2}$$

$$2)\ \lim_{x\to a}\frac{x^3-a^3}{x^4-a^4}=\lim_{x\to a}\frac{(x-a)(x^2+ax+a^2)}{(x-a)(x+a)(x^2+a^2)}=\frac{a^2+a^2+a^2}{(a+a)(a^2+a^2)}=\frac{3}{4a}$$

$$3)\ \lim_{x\to 0}\frac{5\,\frac{Sin5x}{5x}}{3\,\frac{Sin3x}{3x}}=\frac{5}{3}.\frac{\lim\limits_{5x\to 0}\frac{Sin5x}{5x}}{\lim\limits_{3x\to 0}\frac{Sin3x}{3x}}=\frac{5}{3}$$

$$4)\quad \lim_{y\to 0}\frac{\tan 2y}{3y}=\frac{2}{3}.\lim_{2y\to 0}\frac{Sin2y}{2y}.\lim_{y\to 0}\frac{1}{Cos2y}=\frac{2}{3}$$

$$5)\quad \lim_{x\to 0}\frac{Sin2x}{2x^2+x}=2\lim_{2x\to 0}\frac{Sin2x}{2x}.\lim_{x\to 0}\frac{1}{2x+1}=2$$

$$6)\quad \lim_{x\to\infty}\left(1+Cos\frac{1}{x}\right)=1+Cos0=2$$

$$7)\quad \lim_{x\to\infty}\frac{3x^3+5x^2-7}{10x^3-11x+5}=\lim_{x\to\infty}\frac{3+\frac{5}{x}-\frac{7}{x^3}}{10-\frac{11}{x^2}+\frac{5}{x^3}}=\frac{3}{10}$$

$$8)\quad \lim_{y\to\infty}\frac{3y+7}{y^2-2}=\lim_{y\to\infty}\frac{\frac{3}{y}+\frac{7}{y^2}}{1-\frac{2}{y^2}}=\frac{0}{1}=0$$

$$9)\quad \lim_{x\to\infty}\frac{x^3-1}{2x^2-7x+5}=\lim_{x\to\infty}\frac{1-\frac{1}{x^3}}{\frac{2}{x}-\frac{7}{x^2}+\frac{5}{x^3}}=\frac{1}{0}=\infty$$

$$10)\quad \lim_{x\to -1^-}\frac{1}{x+1}=\frac{1}{-1+1}=-\infty$$

$$11)\quad \lim_{x\to 0}Cos\left(1-\frac{Sinx}{x}\right)=Cos\left(1-\lim_{x\to 0}\frac{Sinx}{x}\right)=Cos0=1$$

$$12)\quad \lim_{x\to 0}Sin\left(\frac{\pi}{2}Cos(\tan x)\right)=Sin\left(\frac{\pi}{2}Cos(\tan 0)\right)=Sin\left(\frac{\pi}{2}Cos0\right)=Sin\frac{\pi}{2}=1$$

EX-5- **Test continuity for the following function :**

$$f(x)=\begin{cases} x^2-1 & -1\le x<0\\ 2x & 0\le x<1\\ 1 & x=1\\ -2x+4 & 1<x\le 2\\ 0 & 2<x\le 3\end{cases}$$

Sol.- **We test the continuity at midpoints $x = 0 , 1 , 2$ and endpoints $x = -1 , 3$.**

$$At \quad x=0 \Rightarrow \qquad f(0)=2*0=0$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}(x^2-1)=-1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0} 2x=0 \neq \lim_{x\to 0^-} f(x)$$

$$Since \quad \lim_{x\to 0} f(x) \ doesn't \ exist$$

$$Hence \quad the \ function \ discontinuous \ at \ x=0$$

$$At \quad x=1 \Rightarrow \quad f(1)=1$$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1} 2x=2$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(-2x+4)=2=\lim_{x\to 1^-} f(x)=\lim_{x\to 1} f(x)$$

$$Since \quad \lim_{x\to 1} f(x) \neq f(1)$$

$$Hence \quad the \ function \ is \ discontinuous \ at \ x=1$$

$$At \quad x=2 \Rightarrow \quad f(2)=-2*2+4=0$$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2}(-2x+4)=0$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2} 0=0=\lim_{x\to 2^-} f(x)=\lim_{x\to 2} f(x)$$

$$Since \quad \lim_{x\to 2} f(x)=f(2)=0$$

$$Hence \quad the \ function \ is \ continuous \ at \ x=2$$

$$At \quad x=-1 \Rightarrow \quad f(-1)=(-1)^2-1=0$$

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1}(x^2-1)=0=f(-1)$$

$$Hence \quad the \ function \ is \ continuous \ at \ x=-1$$

$$At \quad x=3 \Rightarrow \quad f(3)=0$$

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3} 0=0=f(3)$$

$$Hence \quad the \ function \ is \ continuous \ at \ x=3$$

<u>*EX-6*</u>- What value should be assigned to a to make the function :

$$f(x)=\begin{Bmatrix} x^2-1 & x<3 \\ 2ax & x\geq 3 \end{Bmatrix}$$ **continuous at $x = 3$?**

***<u>Sol.</u>* –**

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x) \Rightarrow \lim_{x\to 3}(x^2-1)=\lim_{x\to 3} 2ax \Rightarrow \quad 8=6a \Rightarrow \quad a=\frac{4}{3}$$

Problems (2)

1. Discuss the continuity of :

$$f(x)=\begin{cases} x+\dfrac{1}{x} & for \quad x<0 \\ -x^3 & for \quad 0\le x<1 \\ -1 & for \quad 1\le x<2 \\ 1 & for \quad x=2 \\ 0 & for \quad x>2 \end{cases}$$

(ans.: discontinuous at x=0,2 ; continuous at x=1)

2. Evaluate the following limits :

$a)\ \lim\limits_{x\to\infty}\dfrac{x+Sinx}{2x+5}$ $\qquad$ $b)\ \lim\limits_{x\to\infty}\dfrac{1+Sinx}{x}$

$c)\ \lim\limits_{x\to 0}\dfrac{x}{tan\,3x}$ $\qquad$ $d)\ \lim\limits_{x\to\infty}\dfrac{x.Sinx}{(x+Sinx)^2}$

$e)\ \lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{1-x}$ $\qquad$ $f)\ \lim\limits_{x\to 1}\dfrac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

$g)\ \lim\limits_{n\to\infty}(\sqrt{n^2+1}-n)$

(ans.:a)1/2, b)0, c)1/3, d)0, e)1/2, f)-1/2√2, g)0)

3. Suppose that : $f(x) = x^3 - 3x^2 - 4x + 12$ **and** $h(x)=\begin{cases} \dfrac{f(x)}{x-3} & for \ \ x\ne 3 \\ k & for \ \ x=3 \end{cases}$ **.**

Find : a) all zeros of f **.**

b) the value of k **that makes** h **continuous at** $x=3$ **.**

$(ans.: a)x=\mp 2,3; b)k=5)$