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Mathematics

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Mathematics

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By:

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Title of the course Mathematics
Level : 2^{nd} Stage, 1^{st} Semester
Hours: 2 Hours / Week

References:

1- precalculus   (ninth Edition 2012) by Michael Sullivan.

9.1 Polar Coordinates

PREPARING FOR THIS SECTION *Before getting started, review the following:*

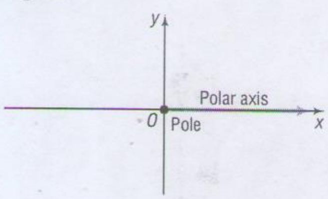
- Rectangular Coordinates (Section 1.1, pp. 2–6)
- Definition of the Trigonometric Functions (Section 6.2, pp. 363–366)
- Inverse Tangent Function (Section 7.1, pp. 442–444)
- Completing the Square (Appendix A, Section A.3, pp. A29–A30)

Now Work the 'Are You Prepared?' problems on page 565.

OBJECTIVES
1. Plot Points Using Polar Coordinates (p. 558)
2. Convert from Polar Coordinates to Rectangular Coordinates (p. 560)
3. Convert from Rectangular Coordinates to Polar Coordinates (p. 562)
4. Transform Equations between Polar and Rectangular Forms (p. 564)

So far, we have always used a system of rectangular coordinates to plot points in the plane. Now we are ready to describe another system, called *polar coordinates*. As we shall soon see, in many instances polar coordinates offer certain advantages over rectangular coordinates.

Figure 1



In a rectangular coordinate system, you will recall, a point in the plane is represented by an ordered pair of numbers (x, y), where x and y equal the signed distance of the point from the y-axis and x-axis, respectively. In a polar coordinate system, we select a point, called the **pole,** and then a ray with vertex at the pole, called the **polar axis.** See Figure 1. Comparing the rectangular and polar coordinate systems, we see that the origin in rectangular coordinates coincides with the pole in polar coordinates, and the positive x-axis in rectangular coordinates coincides with the polar axis in polar coordinates.

1 Plot Points Using Polar Coordinates

A point P in a polar coordinate system is represented by an ordered pair of numbers (r, θ). If $r > 0$, then r is the distance of the point from the pole; θ is an angle (in degrees or radians) formed by the polar axis and a ray from the pole through the point. We call the ordered pair (r, θ) the **polar coordinates** of the point. See Figure 2.

As an example, suppose that a point P has polar coordinates $\left(2, \dfrac{\pi}{4}\right)$. We locate P by first drawing an angle of $\dfrac{\pi}{4}$ radian, placing its vertex at the pole and its initial side along the polar axis. Then go out a distance of 2 units along the terminal side of the angle to reach the point P. See Figure 3.

Figure 2

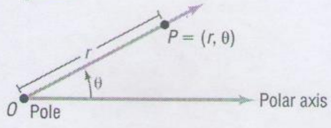
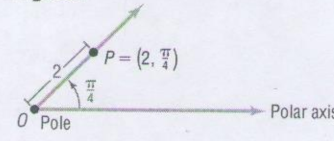


Figure 3



In using polar coordinates (r, θ), it is possible for r to be negative. When this happens, instead of the point being on the terminal side of θ, it is on the ray from the pole extending in the direction *opposite* the terminal side of θ at a distance $|r|$ units from the pole. See Figure 4 for an illustration.

For example, to plot the point $\left(-3, \dfrac{2\pi}{3}\right)$, use the ray in the opposite direction of $\dfrac{2\pi}{3}$ and go out $|-3| = 3$ units along that ray. See Figure 5.

Figure 4

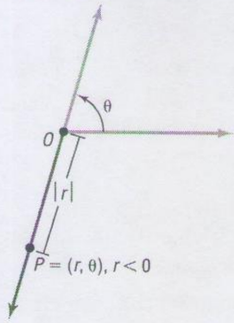
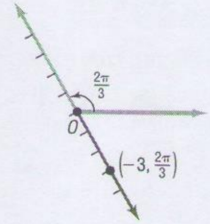


Figure 5



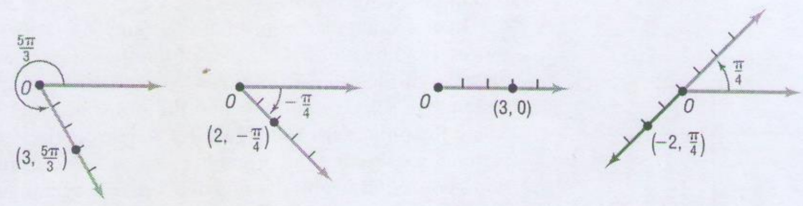
EXAMPLE 1 Plotting Points Using Polar Coordinates

Plot the points with the following polar coordinates:

(a) $\left(3, \frac{5\pi}{3}\right)$ (b) $\left(2, -\frac{\pi}{4}\right)$ (c) $(3, 0)$ (d) $\left(-2, \frac{\pi}{4}\right)$

Solution Figure 6 shows the points.

Figure 6



(a) (b) (c) (d)

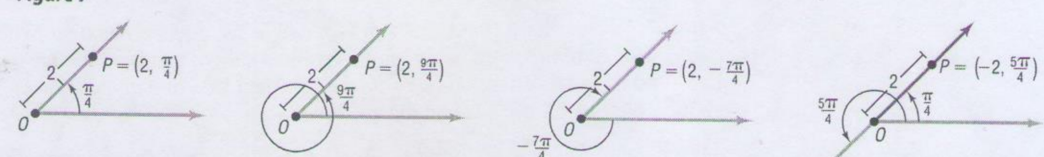
Now Work PROBLEMS 9, 17, AND 27

Recall that an angle measured counterclockwise is positive and an angle measured clockwise is negative. This convention has some interesting consequences relating to polar coordinates.

EXAMPLE 2 Finding Several Polar Coordinates of a Single Point

Consider again the point P with polar coordinates $\left(2, \frac{\pi}{4}\right)$, as shown in Figure 7(a). Because $\frac{\pi}{4}$, $\frac{9\pi}{4}$, and $-\frac{7\pi}{4}$ all have the same terminal side, we also could have located this point P by using the polar coordinates $\left(2, \frac{9\pi}{4}\right)$ or $\left(2, -\frac{7\pi}{4}\right)$, as shown in Figures 7(b) and (c). The point $\left(2, \frac{\pi}{4}\right)$ can also be represented by the polar coordinates $\left(-2, \frac{5\pi}{4}\right)$. See Figure 7(d).

Figure 7



(a) (b) (c) (d)

EXAMPLE 3 **Finding Other Polar Coordinates of a Given Point**

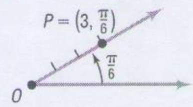
Plot the point P with polar coordinates $\left(3, \frac{\pi}{6}\right)$, and find other polar coordinates (r, θ) of this same point for which:

(a) $r > 0,\quad 2\pi \le \theta < 4\pi$ (b) $r < 0,\quad 0 \le \theta < 2\pi$

(c) $r > 0,\quad -2\pi \le \theta < 0$

Solution The point $\left(3, \frac{\pi}{6}\right)$ is plotted in Figure 8.

Figure 8



(a) Add 1 revolution (2π radians) to the angle $\frac{\pi}{6}$ to get $P = \left(3, \frac{\pi}{6} + 2\pi\right) = \left(3, \frac{13\pi}{6}\right)$. See Figure 9.

(b) Add $\frac{1}{2}$ revolution (π radians) to the angle $\frac{\pi}{6}$ and replace 3 by -3 to get $P = \left(-3, \frac{\pi}{6} + \pi\right) = \left(-3, \frac{7\pi}{6}\right)$. See Figure 10.

(c) Subtract 2π from the angle $\frac{\pi}{6}$ to get $P = \left(3, \frac{\pi}{6} - 2\pi\right) = \left(3, -\frac{11\pi}{6}\right)$. See Figure 11.

Figure 9

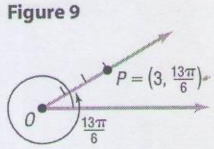


Figure 10

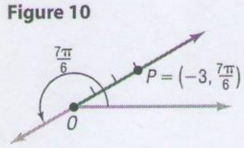
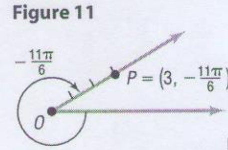


Figure 11



These examples show a major difference between rectangular coordinates and polar coordinates. A point has exactly one pair of rectangular coordinates; however, a point has infinitely many pairs of polar coordinates.

SUMMARY

A point with polar coordinates (r, θ), θ in radians, can also be represented by either of the following:

$$(r, \theta + 2\pi k) \quad \text{or} \quad (-r, \theta + \pi + 2\pi k) \qquad k \text{ any integer}$$

The polar coordinates of the pole are $(0, \theta)$, where θ can be any angle.

Now Work PROBLEM 31

2 Convert from Polar Coordinates to Rectangular Coordinates

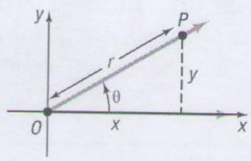
Sometimes we need to convert coordinates or equations in rectangular form to polar form, and vice versa. To do this, recall that the origin in rectangular coordinates is the pole in polar coordinates and that the positive x-axis in rectangular coordinates is the polar axis in polar coordinates.

THEOREM **Conversion from Polar Coordinates to Rectangular Coordinates**

If P is a point with polar coordinates (r, θ), the rectangular coordinates (x, y) of P are given by

$$x = r\cos\theta \qquad y = r\sin\theta \tag{1}$$

Figure 12



Proof Suppose that P has the polar coordinates (r, θ). We seek the rectangular coordinates (x, y) of P. Refer to Figure 12.

If $r = 0$, then, regardless of θ, the point P is the pole, for which the rectangular coordinates are $(0, 0)$. Formula (1) is valid for $r = 0$.

If $r > 0$, the point P is on the terminal side of θ, and $r = d(O, P) = \sqrt{x^2 + y^2}$. Since

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}$$

we have

$$x = r\cos\theta \qquad y = r\sin\theta$$

If $r < 0$ and θ is in radians, the point $P = (r, \theta)$ can be represented as $(-r, \pi + \theta)$, where $-r > 0$. Since

$$\cos(\pi + \theta) = -\cos\theta = \frac{x}{-r} \qquad \sin(\pi + \theta) = -\sin\theta = \frac{y}{-r}$$

we have

$$x = r\cos\theta \qquad y = r\sin\theta$$

■

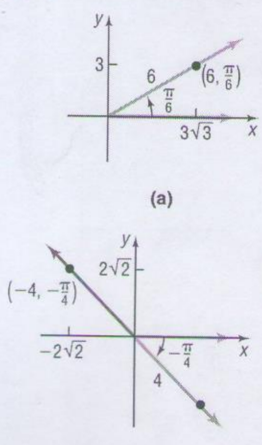
EXAMPLE 4 Converting from Polar Coordinates to Rectangular Coordinates

Find the rectangular coordinates of the points with the following polar coordinates:

(a) $\left(6, \frac{\pi}{6}\right)$ (b) $\left(-4, -\frac{\pi}{4}\right)$

Solution Use formula (1): $x = r\cos\theta$ and $y = r\sin\theta$.

Figure 13



(a)

(b)

(a) Figure 13(a) shows $\left(6, \frac{\pi}{6}\right)$ plotted. Notice that $\left(6, \frac{\pi}{6}\right)$ lies in quadrant I of the rectangular coordinate system. So we expect both the x-coordinate and the y-coordinate to be positive. With $r = 6$ and $\theta = \frac{\pi}{6}$, we have

$$x = r\cos\theta = 6\cos\frac{\pi}{6} = 6\cdot\frac{\sqrt{3}}{2} = 3\sqrt{3}$$

$$y = r\sin\theta = 6\sin\frac{\pi}{6} = 6\cdot\frac{1}{2} = 3$$

The rectangular coordinates of the point $\left(6, \frac{\pi}{6}\right)$ are $\left(3\sqrt{3}, 3\right)$, which lies in quadrant I, as expected.

(b) Figure 13(b) shows $\left(-4, -\frac{\pi}{4}\right)$ plotted. Notice that $\left(-4, -\frac{\pi}{4}\right)$ lies in quadrant II of the rectangular coordinate system. With $r = -4$ and $\theta = -\frac{\pi}{4}$, we have

$$x = r\cos\theta = -4\cos\left(-\frac{\pi}{4}\right) = -4\cdot\frac{\sqrt{2}}{2} = -2\sqrt{2}$$

$$y = r\sin\theta = -4\sin\left(-\frac{\pi}{4}\right) = -4\left(-\frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$$

The rectangular coordinates of the point $\left(-4, -\frac{\pi}{4}\right)$ are $\left(-2\sqrt{2}, 2\sqrt{2}\right)$, which lies in quadrant II, as expected.

COMMENT Many calculators have the capability of converting from polar coordinates to rectangular coordinates. Consult your owner's manual for the proper keystrokes. Since in most cases this procedure is tedious, you will find that using formula (1) is faster. ■

Now Work PROBLEMS 39 AND 51

3 Convert from Rectangular Coordinates to Polar Coordinates

Converting from rectangular coordinates (x, y) to polar coordinates (r, θ) is a little more complicated. Notice that we begin each example by plotting the given rectangular coordinates.

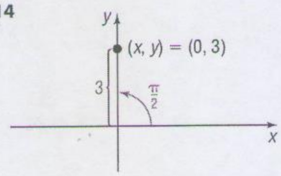
EXAMPLE 5 How to Convert from Rectangular Coordinates to Polar Coordinates with the Point on a Coordinate Axis

Find polar coordinates of a point whose rectangular coordinates are $(0, 3)$.

Step-by-Step Solution

Step 1: Plot the point (x, y) and note the quadrant the point lies in or the coordinate axis the point lies on.

Figure 14



Plot the point $(0, 3)$ in a rectangular coordinate system. See Figure 14. The point lies on the positive y-axis.

Step 2: Determine the distance r from the origin to the point.

The point $(0, 3)$ lies on the y-axis a distance of 3 units from the origin (pole), so $r = 3$.

Step 3: Determine θ.

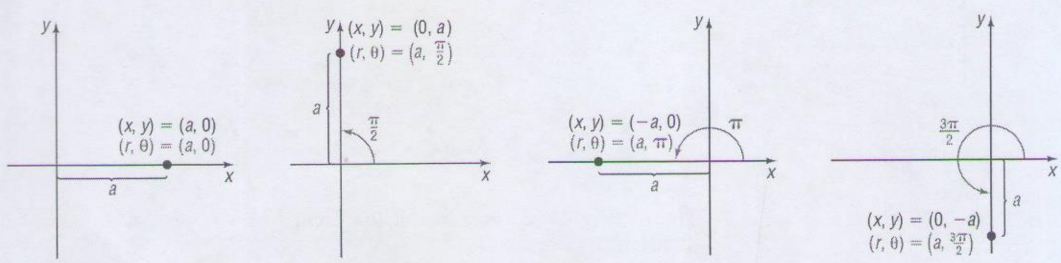
A ray with vertex at the pole through $(0, 3)$ forms an angle $\theta = \frac{\pi}{2}$ with the polar axis.

Polar coordinates for this point can be given by $\left(3, \frac{\pi}{2}\right)$. Other possible representations include $\left(-3, -\frac{\pi}{2}\right)$ and $\left(3, \frac{5\pi}{2}\right)$.

COMMENT Most graphing calculators have the capability of converting from rectangular coordinates to polar coordinates. Consult your owner's manual for the proper keystrokes. ■

Figure 15 shows polar coordinates of points that lie on either the x-axis or the y-axis. In each illustration, $a > 0$.

Figure 15



(a) $(x, y) = (a, 0), a > 0$ (b) $(x, y) = (0, a), a > 0$ (c) $(x, y) = (-a, 0), a > 0$ (d) $(x, y) = (0, -a), a > 0$

Now Work PROBLEM 55

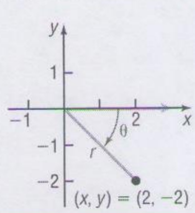
EXAMPLE 6 How to Convert from Rectangular Coordinates to Polar Coordinates with the Point in a Quadrant

Find the polar coordinates of a point whose rectangular coordinates are $(2, -2)$.

Step-by-Step Solution

Step 1: Plot the point (x, y) and note the quadrant the point lies in or the coordinate axis the point lies on.

Figure 16



Plot the point $(2, -2)$ in a rectangular coordinate system. See Figure 16. The point lies in quadrant IV.

Step 2: Determine the distance r from the origin to the point using $r = \sqrt{x^2 + y^2}$.

$$r = \sqrt{x^2 + y^2} = \sqrt{(2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

Step 3: Determine θ.

Find θ by recalling that $\tan\theta = \frac{y}{x}$, so $\theta = \tan^{-1}\frac{y}{x}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Since $(2, -2)$ lies in quadrant IV, we know that $-\frac{\pi}{2} < \theta < 0$. As a result,

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\left(\frac{-2}{2}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

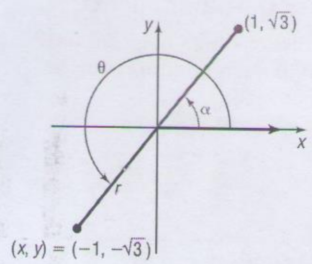
A set of polar coordinates for the point $(2, -2)$ is $\left(2\sqrt{2}, -\frac{\pi}{4}\right)$. Other possible representations include $\left(2\sqrt{2}, \frac{7\pi}{4}\right)$ and $\left(-2\sqrt{2}, \frac{3\pi}{4}\right)$.

EXAMPLE 7 Converting from Rectangular Coordinates to Polar Coordinates

Find polar coordinates of a point whose rectangular coordinates are $\left(-1, -\sqrt{3}\right)$.

Solution

Figure 17



Step 1: See Figure 17. The point lies in quadrant III.

Step 2: The distance r from the origin to the point $\left(-1, -\sqrt{3}\right)$ is

$$r = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = \sqrt{4} = 2$$

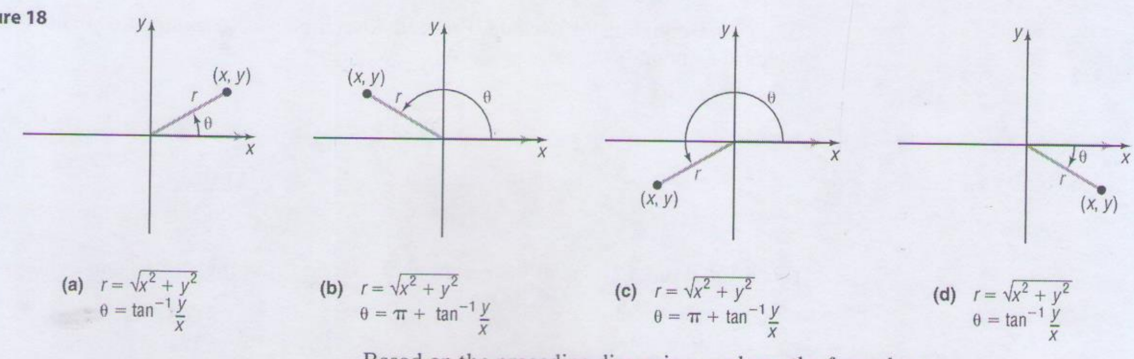
Step 3: To find θ, use $\alpha = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{-\sqrt{3}}{-1} = \tan^{-1}\sqrt{3}$, $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$. Since the point $\left(-1, -\sqrt{3}\right)$ lies in quadrant III and the inverse tangent function gives an angle in quadrant I, add π to the result to obtain an angle in quadrant III.

$$\theta = \pi + \tan^{-1}\sqrt{3} = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

A set of polar coordinates for this point is $\left(2, \frac{4\pi}{3}\right)$. Other possible representations include $\left(-2, \frac{\pi}{3}\right)$ and $\left(2, -\frac{2\pi}{3}\right)$.

Figure 18 shows how to find polar coordinates of a point that lies in a quadrant when its rectangular coordinates (x, y) are given.

Figure 18



(a) $r = \sqrt{x^2 + y^2}$
$\theta = \tan^{-1}\frac{y}{x}$

(b) $r = \sqrt{x^2 + y^2}$
$\theta = \pi + \tan^{-1}\frac{y}{x}$

(c) $r = \sqrt{x^2 + y^2}$
$\theta = \pi + \tan^{-1}\frac{y}{x}$

(d) $r = \sqrt{x^2 + y^2}$
$\theta = \tan^{-1}\frac{y}{x}$

Based on the preceding discussion, we have the formulas

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \quad \text{if } x \neq 0 \tag{2}$$

To use formula (2) effectively, follow these steps:

Steps for Converting from Rectangular to Polar Coordinates

STEP 1: Always plot the point (x, y) first, as we did in Examples 5, 6, and 7. Note the quadrant the point lies in or the coordinate axis the point lies on.

STEP 2: If $x = 0$ or $y = 0$, use your illustration to find r. If $x \neq 0$ and $y \neq 0$, then $r = \sqrt{x^2 + y^2}$.

STEP 3: Find θ. If $x = 0$ or $y = 0$, use your illustration to find θ. If $x \neq 0$ and $y \neq 0$, note the quadrant in which the point lies.

$$\text{Quadrant I or IV:} \quad \theta = \tan^{-1}\frac{y}{x}$$

$$\text{Quadrant II or III:} \quad \theta = \pi + \tan^{-1}\frac{y}{x}$$

Now Work PROBLEM 59

4 Transform Equations between Polar and Rectangular Forms

Formulas (1) and (2) may also be used to transform equations from polar form to rectangular form, and vice versa. Two common techniques for transforming an equation from polar form to rectangular form are the following:

1. Multiplying both sides of the equation by r
2. Squaring both sides of the equation

EXAMPLE 8 **Transforming an Equation from Polar to Rectangular Form**

Transform the equation $r = 6\cos\theta$ from polar coordinates to rectangular coordinates, and identify the graph.

Solution If we multiply each side by r, it will be easier to apply formulas (1) and (2).

$$\begin{aligned} r &= 6\cos\theta \\ r^2 &= 6r\cos\theta && \text{Multiply each side by } r. \\ x^2 + y^2 &= 6x && r^2 = x^2 + y^2;\ x = r\cos\theta \end{aligned}$$

This is the equation of a circle. Proceed to complete the square to obtain the standard form of the equation.

$$\begin{aligned} x^2 + y^2 &= 6x \\ (x^2 - 6x) + y^2 &= 0 && \text{General form} \\ (x^2 - 6x + 9) + y^2 &= 9 && \text{Complete the square in } x. \\ (x - 3)^2 + y^2 &= 9 && \text{Factor.} \end{aligned}$$

This is the standard form of the equation of a circle with center $(3, 0)$ and radius 3.

Now Work PROBLEM 75

EXAMPLE 9 **Transforming an Equation from Rectangular to Polar Form**

Transform the equation $4xy = 9$ from rectangular coordinates to polar coordinates.

Solution Use formula (1): $x = r\cos\theta$ and $y = r\sin\theta$.

$$4xy = 9$$
$$4(r\cos\theta)(r\sin\theta) = 9 \quad x = r\cos\theta,\ y = r\sin\theta$$
$$4r^2\cos\theta\sin\theta = 9$$

This is the polar form of the equation. It can be simplified as shown next:

$$2r^2(2\sin\theta\cos\theta) = 9 \quad \text{Factor out } 2r^2.$$
$$2r^2\sin(2\theta) = 9 \quad \text{Double-angle Formula}$$

Now Work PROBLEM 69

9.1 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Plot the point whose rectangular coordinates are $(3, -1)$. What quadrant does the point lie in? (pp. 2–6)
2. To complete the square of $x^2 + 6x$, add ________. (pp. A29–A30)
3. If $P = (a, b)$ is a point on the terminal side of the angle θ at a distance r from the origin, then $\tan\theta =$ ____. (pp. 363–366)
4. $\tan^{-1}(-1) =$ ________. (pp. 442–444)

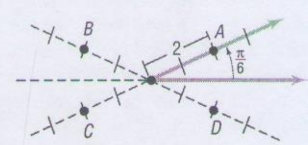
Concepts and Vocabulary

5. The origin in rectangular coordinates coincides with the ______ in polar coordinates; the positive x-axis in rectangular coordinates coincides with the ______ ______ in polar coordinates.
6. ***True or False*** In the polar coordinates (r, θ), r can be negative.
7. ***True or False*** The polar coordinates of a point are unique.
8. If P is a point with polar coordinates (r, θ), the rectangular coordinates (x, y) of P are given by $x =$ _______ and $y =$ _______.

Skill Building

In Problems 9–16, match each point in polar coordinates with either A, B, C, or D on the graph.

9. $\left(2, -\frac{11\pi}{6}\right)$
10. $\left(-2, -\frac{\pi}{6}\right)$
11. $\left(-2, \frac{\pi}{6}\right)$
12. $\left(2, \frac{7\pi}{6}\right)$
13. $\left(2, \frac{5\pi}{6}\right)$
14. $\left(-2, \frac{5\pi}{6}\right)$
15. $\left(-2, \frac{7\pi}{6}\right)$
16. $\left(2, \frac{11\pi}{6}\right)$



In Problems 17–30, plot each point given in polar coordinates.

17. $(3, 90°)$
18. $(4, 270°)$
19. $(-2, 0)$
20. $(-3, \pi)$
21. $\left(6, \frac{\pi}{6}\right)$
22. $\left(5, \frac{5\pi}{3}\right)$
23. $(-2, 135°)$
24. $(-3, 120°)$
25. $\left(4, -\frac{2\pi}{3}\right)$
26. $\left(2, -\frac{5\pi}{4}\right)$
27. $\left(-1, -\frac{\pi}{3}\right)$
28. $\left(-3, -\frac{3\pi}{4}\right)$
29. $(-2, -\pi)$
30. $\left(-3, -\frac{\pi}{2}\right)$

In Problems 31–38, plot each point given in polar coordinates, and find other polar coordinates (r, θ) of the point for which:

(a) $r > 0,\ -2\pi \le \theta < 0$ (b) $r < 0,\ 0 \le \theta < 2\pi$ (c) $r > 0,\ 2\pi \le \theta < 4\pi$

31. $\left(5, \frac{2\pi}{3}\right)$
32. $\left(4, \frac{3\pi}{4}\right)$
33. $(-2, 3\pi)$
34. $(-3, 4\pi)$
35. $\left(1, \frac{\pi}{2}\right)$
36. $(2, \pi)$
37. $\left(-3, -\frac{\pi}{4}\right)$
38. $\left(-2, -\frac{2\pi}{3}\right)$

Explaining Concepts: Discussion and Writing

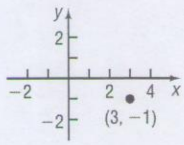
85. In converting from polar coordinates to rectangular coordinates, what formulas will you use?

86. Explain how you proceed to convert from rectangular coordinates to polar coordinates.

87. Is the street system in your town based on a rectangular coordinate system, a polar coordinate system, or some other system? Explain.

'Are You Prepared?' Answers

1.



; quadrant IV **2.** 9 **3.** $\frac{b}{a}$ **4.** $-\frac{\pi}{4}$

9.2 Polar Equations and Graphs

PREPARING FOR THIS SECTION *Before getting started, review the following:*

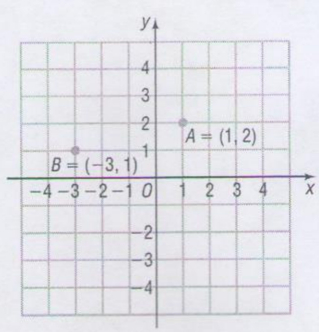
- Symmetry (Section 1.2, pp. 12–14)
- Circles (Section 1.4, pp. 34–37)
- Even–Odd Properties of Trigonometric Functions (Section 6.3, p. 389)
- Difference Formulas for Sine and Cosine (Section 7.4, pp. 472 and 475)
- Values of the Sine and Cosine Functions at Certain Angles (Section 6.2, pp. 366–375)

Now Work the 'Are You Prepared?' problems on page 579.

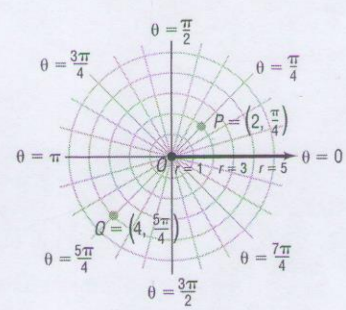
OBJECTIVES
1 Identify and Graph Polar Equations by Converting to Rectangular Equations (p. 568)
2 Test Polar Equations for Symmetry (p. 571)
3 Graph Polar Equations by Plotting Points (p. 572)

Just as a rectangular grid may be used to plot points given by rectangular coordinates, as in Figure 19(a), we can use a grid consisting of concentric circles (with centers at the pole) and rays (with vertices at the pole) to plot points given by polar coordinates, as shown in Figure 19(b). We use such **polar grids** to graph *polar equations*.

Figure 19



(a) Rectangular grid



(b) Polar grid

DEFINITION An equation whose variables are polar coordinates is called a **polar equation.** The **graph of a polar equation** consists of all points whose polar coordinates satisfy the equation.

1 Identify and Graph Polar Equations by Converting to Rectangular Equations

One method used to graph a polar equation is to convert the equation to rectangular coordinates. In the discussion that follows, (x, y) represent the rectangular coordinates of a point P, and (r, θ) represent polar coordinates of the point P.

EXAMPLE 1 Identifying and Graphing a Polar Equation (Circle)

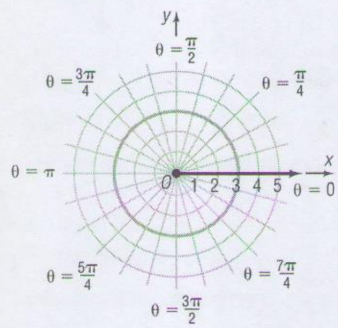
Identify and graph the equation: $r = 3$

Solution Convert the polar equation to a rectangular equation.

$$\begin{aligned} r &= 3 \\ r^2 &= 9 && \text{Square both sides.} \\ x^2 + y^2 &= 9 && r^2 = x^2 + y^2 \end{aligned}$$

The graph of $r = 3$ is a circle, with center at the pole and radius 3. See Figure 20.

Figure 20
$r = 3$ or $x^2 + y^2 = 9$



Now Work PROBLEM 13

EXAMPLE 2 Identifying and Graphing a Polar Equation (Line)

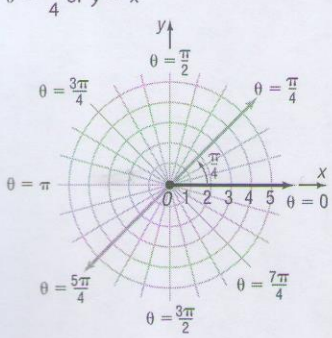
Identify and graph the equation: $\theta = \dfrac{\pi}{4}$

Solution Convert the polar equation to a rectangular equation.

$$\begin{aligned} \theta &= \frac{\pi}{4} \\ \tan\theta &= \tan\frac{\pi}{4} && \text{Take the tangent of both sides.} \\ \frac{y}{x} &= 1 && \tan\theta = \frac{y}{x};\ \tan\frac{\pi}{4} = 1 \\ y &= x \end{aligned}$$

The graph of $\theta = \dfrac{\pi}{4}$ is a line passing through the pole making an angle of $\dfrac{\pi}{4}$ with the polar axis. See Figure 21.

Figure 21
$\theta = \dfrac{\pi}{4}$ or $y = x$



Now Work PROBLEM 15

EXAMPLE 3 **Identifying and Graphing a Polar Equation (Horizontal Line)**

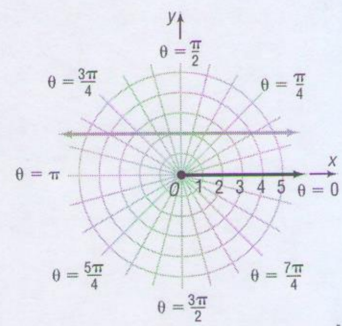
Identify and graph the equation: $r \sin\theta = 2$

Solution Since $y = r\sin\theta$, we can write the equation as

$$y = 2$$

We conclude that the graph of $r\sin\theta = 2$ is a horizontal line 2 units above the pole. See Figure 22.

Figure 22
$r\sin\theta = 2$ or $y = 2$



COMMENT A graphing utility can be used to graph polar equations. Read Using a Graphing Utility to Graph a Polar Equation, Appendix B, Section B.8. ■

EXAMPLE 4 **Identifying and Graphing a Polar Equation (Vertical Line)**

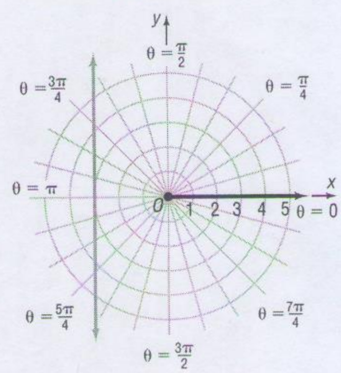
Identify and graph the equation: $r\cos\theta = -3$

Solution Since $x = r\cos\theta$, we can write the equation as

$$x = -3$$

We conclude that the graph of $r\cos\theta = -3$ is a vertical line 3 units to the left of the pole. See Figure 23.

Figure 23
$r\cos\theta = -3$ or $x = -3$



Based on Examples 3 and 4, we are led to the following results. (The proofs are left as exercises. See Problems 81 and 82.)

THEOREM Let a be a real number. Then the graph of the equation

$$\boxed{r\sin\theta = a}$$

is a horizontal line. It lies a units above the pole if $a \geq 0$ and $|a|$ units below the pole if $a < 0$.

The graph of the equation

$$\boxed{r\cos\theta = a}$$

is a vertical line. It lies a units to the right of the pole if $a \geq 0$ and $|a|$ units to the left of the pole if $a < 0$.

Now Work PROBLEM 19

EXAMPLE 5 **Identifying and Graphing a Polar Equation (Circle)**

Identify and graph the equation: $r = 4\sin\theta$

Solution To transform the equation to rectangular coordinates, multiply each side by r.

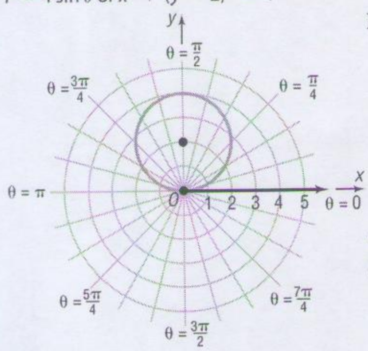
$$r^2 = 4r\sin\theta$$

Now use the facts that $r^2 = x^2 + y^2$ and $y = r\sin\theta$. Then

$$\begin{aligned} x^2 + y^2 &= 4y \\ x^2 + (y^2 - 4y) &= 0 \\ x^2 + (y^2 - 4y + 4) &= 4 && \text{Complete the square in } y. \\ x^2 + (y-2)^2 &= 4 && \text{Factor.} \end{aligned}$$

This is the standard equation of a circle with center at $(0, 2)$ in rectangular coordinates and radius 2. See Figure 24.

Figure 24
$r = 4\sin\theta$ or $x^2 + (y-2)^2 = 4$



EXAMPLE 6 **Identifying and Graphing a Polar Equation (Circle)**

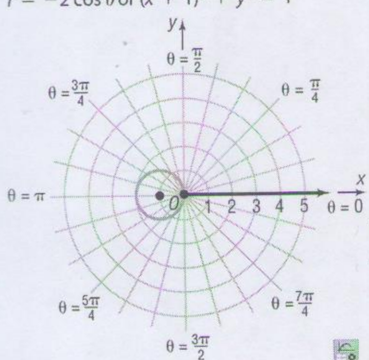
Identify and graph the equation: $r = -2\cos\theta$

Solution Proceed as in Example 5.

$$\begin{aligned} r^2 &= -2r\cos\theta && \text{Multiply both sides by } r. \\ x^2 + y^2 &= -2x && r^2 = x^2 + y^2;\ x = r\cos\theta \\ x^2 + 2x + y^2 &= 0 \\ (x^2 + 2x + 1) + y^2 &= 1 && \text{Complete the square in } x. \\ (x+1)^2 + y^2 &= 1 && \text{Factor.} \end{aligned}$$

This is the standard equation of a circle with center at $(-1, 0)$ in rectangular coordinates and radius 1. See Figure 25.

Figure 25
$r = -2\cos\theta$ or $(x+1)^2 + y^2 = 1$



Exploration

Using a square screen, graph $r_1 = \sin\theta$, $r_2 = 2\sin\theta$, and $r_3 = 3\sin\theta$. Do you see the pattern? Clear the screen and graph $r_1 = -\sin\theta$, $r_2 = -2\sin\theta$, and $r_3 = -3\sin\theta$. Do you see the pattern? Clear the screen and graph $r_1 = \cos\theta$, $r_2 = 2\cos\theta$, and $r_3 = 3\cos\theta$. Do you see the pattern? Clear the screen and graph $r_1 = -\cos\theta$, $r_2 = -2\cos\theta$, and $r_3 = -3\cos\theta$. Do you see the pattern?

Based on Examples 5 and 6 and the preceding Exploration, we are led to the following results. (The proofs are left as exercises. See Problems 83–86.)

THEOREM

Let a be a positive real number. Then

	Equation	Description
(a)	$r = 2a\sin\theta$	Circle: radius a; center at $(0, a)$ in rectangular coordinates
(b)	$r = -2a\sin\theta$	Circle: radius a; center at $(0, -a)$ in rectangular coordinates
(c)	$r = 2a\cos\theta$	Circle: radius a; center at $(a, 0)$ in rectangular coordinates
(d)	$r = -2a\cos\theta$	Circle: radius a; center at $(-a, 0)$ in rectangular coordinates

Each circle passes through the pole.

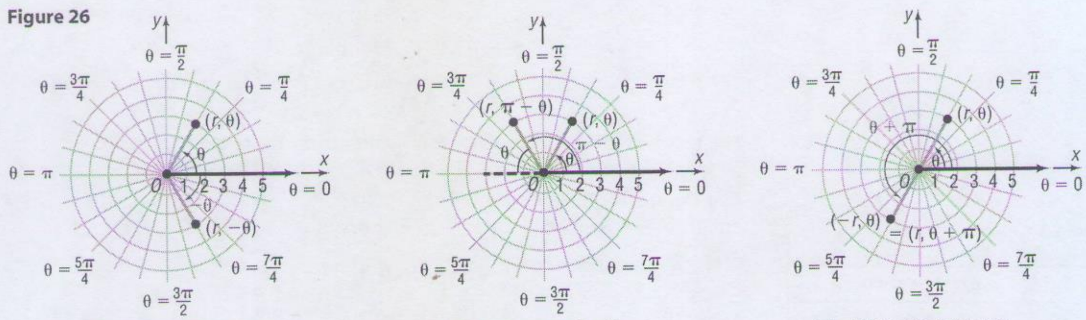
Now Work PROBLEM 21

The method of converting a polar equation to an identifiable rectangular equation to obtain the graph is not always helpful, nor is it always necessary. Usually, we set up a table that lists several points on the graph. By checking for symmetry, it may be possible to reduce the number of points needed to draw the graph.

2 Test Polar Equations for Symmetry

In polar coordinates, the points (r, θ) and $(r, -\theta)$ are symmetric with respect to the polar axis (and to the x-axis). See Figure 26(a). The points (r, θ) and $(r, \pi - \theta)$ are symmetric with respect to the line $\theta = \frac{\pi}{2}$ (the y-axis). See Figure 26(b). The points (r, θ) and $(-r, \theta)$ are symmetric with respect to the pole (the origin). See Figure 26(c).

Figure 26



(a) Points symmetric with respect to the polar axis

(b) Points symmetric with respect to the line $\theta = \frac{\pi}{2}$

(c) Points symmetric with respect to the pole

The following tests are a consequence of these observations.

THEOREM

Tests for Symmetry

Symmetry with Respect to the Polar Axis (x-Axis)

In a polar equation, replace θ by $-\theta$. If an equivalent equation results, the graph is symmetric with respect to the polar axis.

Symmetry with Respect to the Line $\theta = \frac{\pi}{2}$ (y-Axis)

In a polar equation, replace θ by $\pi - \theta$. If an equivalent equation results, the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

Symmetry with Respect to the Pole (Origin)

In a polar equation, replace r by $-r$ or θ by $\theta + \pi$. If an equivalent equation results, the graph is symmetric with respect to the pole.

The three tests for symmetry given here are *sufficient* conditions for symmetry, but they are not *necessary* conditions. That is, an equation may fail these tests and still have a graph that is symmetric with respect to the polar axis, the line $\theta = \frac{\pi}{2}$, or the pole. For example, the graph of $r = \sin(2\theta)$ turns out to be symmetric with respect to the polar axis, the line $\theta = \frac{\pi}{2}$, and the pole, but only the test for symmetry with respect to the pole (replace θ by $\theta + \pi$) works. See also Problems 87–89.

3 Graph Polar Equations by Plotting Points

EXAMPLE 7 **Graphing a Polar Equation (Cardioid)**

Graph the equation: $r = 1 - \sin\theta$

Solution Check for symmetry first.

Polar Axis: Replace θ by $-\theta$. The result is

$$r = 1 - \sin(-\theta) = 1 + \sin\theta \quad \sin(-\theta) = -\sin\theta$$

The test fails, so the graph may or may not be symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: Replace θ by $\pi - \theta$. The result is

$$\begin{aligned} r = 1 - \sin(\pi - \theta) &= 1 - (\sin\pi\cos\theta - \cos\pi\sin\theta) \\ &= 1 - [0\cdot\cos\theta - (-1)\sin\theta] = 1 - \sin\theta \end{aligned}$$

The test is satisfied, so the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The Pole: Replace r by $-r$. Then the result is $-r = 1 - \sin\theta$, so $r = -1 + \sin\theta$. The test fails. Replace θ by $\theta + \pi$. The result is

$$\begin{aligned} r &= 1 - \sin(\theta + \pi) \\ &= 1 - [\sin\theta\cos\pi + \cos\theta\sin\pi] \\ &= 1 - [\sin\theta\cdot(-1) + \cos\theta\cdot 0] \\ &= 1 + \sin\theta \end{aligned}$$

This test also fails. So the graph may or may not be symmetric with respect to the pole.

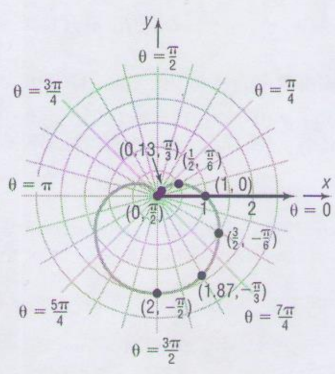
Next, identify points on the graph by assigning values to the angle θ and calculating the corresponding values of r. Due to the periodicity of the sine function and the symmetry with respect to the line $\theta = \frac{\pi}{2}$, we only need to assign values to θ from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, as given in Table 1.

Table 1

θ	$r = 1 - \sin\theta$
$-\frac{\pi}{2}$	$1 - (-1) = 2$
$-\frac{\pi}{3}$	$1 - \left(-\frac{\sqrt{3}}{2}\right) \approx 1.87$
$-\frac{\pi}{6}$	$1 - \left(-\frac{1}{2}\right) = \frac{3}{2}$
0	$1 - 0 = 1$
$\frac{\pi}{6}$	$1 - \frac{1}{2} = \frac{1}{2}$
$\frac{\pi}{3}$	$1 - \frac{\sqrt{3}}{2} \approx 0.13$
$\frac{\pi}{2}$	$1 - 1 = 0$

Now plot the points (r, θ) from Table 1 and trace out the graph, beginning at the point $\left(2, -\frac{\pi}{2}\right)$ and ending at the point $\left(0, \frac{\pi}{2}\right)$. Then reflect this portion of the graph about the line $\theta = \frac{\pi}{2}$ (the y-axis) to obtain the complete graph. See Figure 27.

Figure 27
$r = 1 - \sin\theta$



The curve in Figure 27 is an example of a *cardioid* (a heart-shaped curve).

Exploration

Graph $r_1 = 1 + \sin\theta$. Clear the screen and graph $r_1 = 1 - \cos\theta$. Clear the screen and graph $r_1 = 1 + \cos\theta$. Do you see a pattern?

DEFINITION

Cardioids are characterized by equations of the form

$$r = a(1 + \cos\theta) \qquad r = a(1 + \sin\theta)$$
$$r = a(1 - \cos\theta) \qquad r = a(1 - \sin\theta)$$

where $a > 0$. The graph of a cardioid passes through the pole.

Now Work PROBLEM 37

EXAMPLE 8 **Graphing a Polar Equation (Limaçon without an Inner Loop)**

Graph the equation: $r = 3 + 2\cos\theta$

Solution Check for symmetry first.

Polar Axis: Replace θ by $-\theta$. The result is

$$r = 3 + 2\cos(-\theta) = 3 + 2\cos\theta \quad \cos(-\theta) = \cos\theta$$

The test is satisfied, so the graph is symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: Replace θ by $\pi - \theta$. The result is

$$r = 3 + 2\cos(\pi - \theta) = 3 + 2(\cos\pi\cos\theta + \sin\pi\sin\theta)$$
$$= 3 - 2\cos\theta$$

The test fails, so the graph may or may not be symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The Pole: Replace r by $-r$. The test fails, so the graph may or may not be symmetric with respect to the pole. Replace θ by $\theta + \pi$. The test fails, so the graph may or may not be symmetric with respect to the pole.

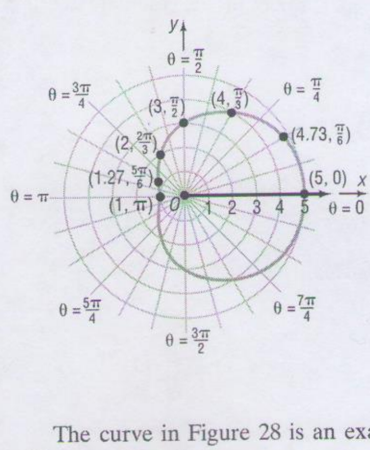
Next, identify points on the graph by assigning values to the angle θ and calculating the corresponding values of r. Due to the periodicity of the cosine function and the symmetry with respect to the polar axis, we only need to assign values to θ from 0 to π, as given in Table 2.

Table 2

θ	$r = 3 + 2\cos\theta$
0	$3 + 2(1) = 5$
$\frac{\pi}{6}$	$3 + 2\left(\frac{\sqrt{3}}{2}\right) \approx 4.73$
$\frac{\pi}{3}$	$3 + 2\left(\frac{1}{2}\right) = 4$
$\frac{\pi}{2}$	$3 + 2(0) = 3$
$\frac{2\pi}{3}$	$3 + 2\left(-\frac{1}{2}\right) = 2$
$\frac{5\pi}{6}$	$3 + 2\left(-\frac{\sqrt{3}}{2}\right) \approx 1.27$
π	$3 + 2(-1) = 1$

Now plot the points (r, θ) from Table 2 and trace out the graph, beginning at the point $(5, 0)$ and ending at the point $(1, \pi)$. Then reflect this portion of the graph about the polar axis (the x-axis) to obtain the complete graph. See Figure 28.

Figure 28
$r = 3 + 2\cos\theta$



The curve in Figure 28 is an example of a *limaçon* (a French word for *snail*) *without an inner loop*.

Exploration

Graph $r_1 = 3 - 2\cos\theta$. Clear the screen and graph $r_1 = 3 + 2\sin\theta$. Clear the screen and graph $r_1 = 3 - 2\sin\theta$. Do you see a pattern?

DEFINITION

Limaçons without an inner loop are characterized by equations of the form

$$r = a + b\cos\theta \qquad r = a + b\sin\theta$$
$$r = a - b\cos\theta \qquad r = a - b\sin\theta$$

where $a > 0$, $b > 0$, and $a > b$. The graph of a limaçon without an inner loop does not pass through the pole.

Now Work PROBLEM 43

EXAMPLE 9 Graphing a Polar Equation (Limaçon with an Inner Loop)

Graph the equation: $r = 1 + 2\cos\theta$

Solution First, check for symmetry.

Polar Axis: Replace θ by $-\theta$. The result is

$$r = 1 + 2\cos(-\theta) = 1 + 2\cos\theta$$

The test is satisfied, so the graph is symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: Replace θ by $\pi - \theta$. The result is

$$r = 1 + 2\cos(\pi - \theta) = 1 + 2(\cos\pi\cos\theta + \sin\pi\sin\theta)$$
$$= 1 - 2\cos\theta$$

The test fails, so the graph may or may not be symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The Pole: Replace r by $-r$. The test fails, so the graph may or may not be symmetric with respect to the pole. Replace θ by $\theta + \pi$. The test fails, so the graph may or may not be symmetric with respect to the pole.

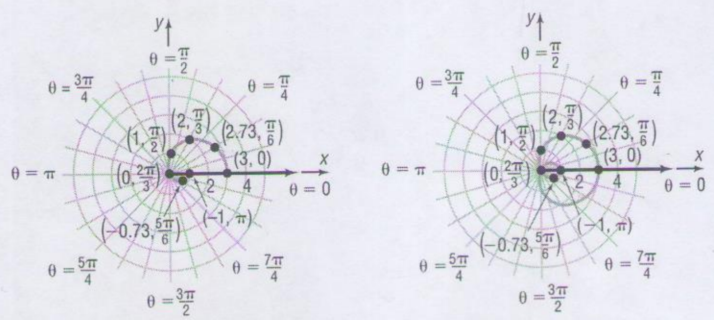
Next, identify points on the graph of $r = 1 + 2\cos\theta$ by assigning values to the angle θ and calculating the corresponding values of r. Due to the periodicity of the cosine function and the symmetry with respect to the polar axis, we only need to assign values to θ from 0 to π, as given in Table 3.

Now plot the points (r, θ) from Table 3, beginning at $(3, 0)$ and ending at $(-1, \pi)$. See Figure 29(a). Finally, reflect this portion of the graph about the polar axis (the x-axis) to obtain the complete graph. See Figure 29(b).

Table 3

θ	$r = 1 + 2\cos\theta$
0	$1 + 2(1) = 3$
$\frac{\pi}{6}$	$1 + 2\left(\frac{\sqrt{3}}{2}\right) \approx 2.73$
$\frac{\pi}{3}$	$1 + 2\left(\frac{1}{2}\right) = 2$
$\frac{\pi}{2}$	$1 + 2(0) = 1$
$\frac{2\pi}{3}$	$1 + 2\left(-\frac{1}{2}\right) = 0$
$\frac{5\pi}{6}$	$1 + 2\left(-\frac{\sqrt{3}}{2}\right) \approx -0.73$
π	$1 + 2(-1) = -1$

Figure 29



(a)

(b) $r = 1 + 2\cos\theta$

The curve in Figure 29(b) is an example of a *limaçon with an inner loop*.

Exploration

Graph $r_1 = 1 - 2\cos\theta$. Clear the screen and graph $r_1 = 1 + 2\sin\theta$. Clear the screen and graph $r_1 = 1 - 2\sin\theta$. Do you see a pattern?

DEFINITION

Limaçons with an inner loop are characterized by equations of the form

$$r = a + b\cos\theta \qquad r = a + b\sin\theta$$
$$r = a - b\cos\theta \qquad r = a - b\sin\theta$$

where $a > 0$, $b > 0$, and $a < b$. The graph of a limaçon with an inner loop will pass through the pole twice.

Now Work PROBLEM 45

EXAMPLE 10 **Graphing a Polar Equation (Rose)**

Graph the equation: $r = 2\cos(2\theta)$

Solution Check for symmetry.

Polar Axis: If we replace θ by $-\theta$, the result is

$$r = 2\cos[2(-\theta)] = 2\cos(2\theta)$$

The test is satisfied, so the graph is symmetric with respect to the polar axis.

The Line $\theta = \frac{\pi}{2}$: If we replace θ by $\pi - \theta$, we obtain

$$r = 2\cos[2(\pi - \theta)] = 2\cos(2\pi - 2\theta) = 2\cos(2\theta)$$

The test is satisfied, so the graph is symmetric with respect to the line $\theta = \frac{\pi}{2}$.

The Pole: Since the graph is symmetric with respect to both the polar axis and the line $\theta = \frac{\pi}{2}$, it must be symmetric with respect to the pole.

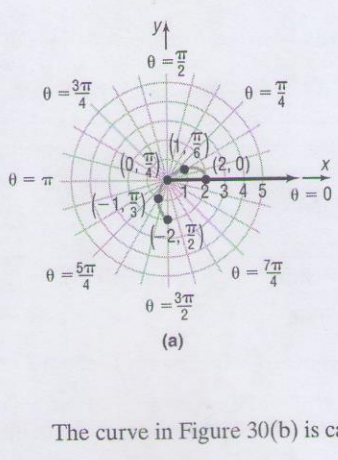
Table 4

θ	$r = 2\cos(2\theta)$
0	$2(1) = 2$
$\frac{\pi}{6}$	$2\left(\frac{1}{2}\right) = 1$
$\frac{\pi}{4}$	$2(0) = 0$
$\frac{\pi}{3}$	$2\left(-\frac{1}{2}\right) = -1$
$\frac{\pi}{2}$	$2(-1) = -2$

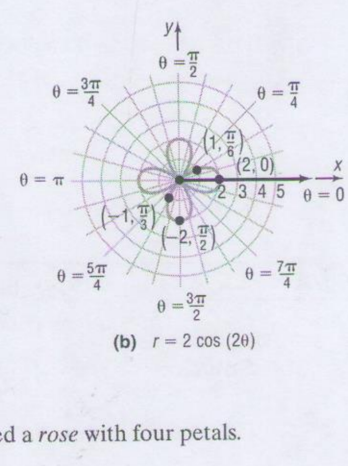
Next, construct Table 4. Due to the periodicity of the cosine function and the symmetry with respect to the polar axis, the line $\theta = \frac{\pi}{2}$, and the pole, we consider only values of θ from 0 to $\frac{\pi}{2}$.

Plot and connect these points in Figure 30(a). Finally, because of symmetry, reflect this portion of the graph first about the polar axis (the x-axis) and then about the line $\theta = \frac{\pi}{2}$ (the y-axis) to obtain the complete graph. See Figure 30(b).

Figure 30



(a)



(b) $r = 2\cos(2\theta)$

Exploration

Graph $r_1 = 2\cos(4\theta)$; clear the screen and graph $r_1 = 2\cos(6\theta)$. How many petals did each of these graphs have?

Clear the screen and graph, in order, each on a clear screen, $r_1 = 2\cos(3\theta)$, $r_1 = 2\cos(5\theta)$, and $r_1 = 2\cos(7\theta)$. What do you notice about the number of petals?

The curve in Figure 30(b) is called a *rose* with four petals.

DEFINITION

Rose curves are characterized by equations of the form

$$r = a\cos(n\theta), \qquad r = a\sin(n\theta), \qquad a \neq 0$$

and have graphs that are rose shaped. If $n \neq 0$ is even, the rose has $2n$ petals; if $n \neq \pm 1$ is odd, the rose has n petals.

Now Work PROBLEM 49

EXAMPLE 11 Graphing a Polar Equation (Lemniscate)

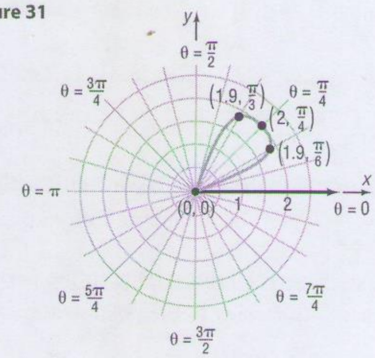
Graph the equation: $r^2 = 4\sin(2\theta)$

Solution We leave it to you to verify that the graph is symmetric with respect to the pole. Because of the symmetry with respect to the pole, we only need to consider values of θ between $\theta = 0$ and $\theta = \pi$. Note that there are no points on the graph for $\frac{\pi}{2} < \theta < \pi$ (quadrant II), since $r^2 < 0$ for such values. Table 5 lists points on the graph for values of $\theta = 0$ through $\theta = \frac{\pi}{2}$. The points from Table 5 where $r \geq 0$ are plotted in Figure 31(a). The remaining points on the graph may be obtained by using symmetry. Figure 31(b) shows the final graph drawn.

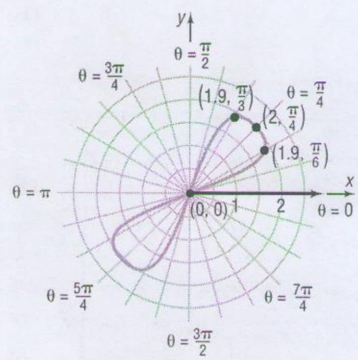
Table 5

θ	$r^2 = 4\sin(2\theta)$	r
0	$4(0) = 0$	0
$\frac{\pi}{6}$	$4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$	± 1.9
$\frac{\pi}{4}$	$4(1) = 4$	± 2
$\frac{\pi}{3}$	$4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$	± 1.9
$\frac{\pi}{2}$	$4(0) = 0$	0

Figure 31



(a)



(b) $r^2 = 4\sin(2\theta)$

The curve in Figure 31(b) is an example of a *lemniscate* (from the Greek word *ribbon*).

DEFINITION

Lemniscates are characterized by equations of the form

$$r^2 = a^2\sin(2\theta) \qquad r^2 = a^2\cos(2\theta)$$

where $a \neq 0$, and have graphs that are propeller shaped.

Now Work PROBLEM 53

EXAMPLE 12 Graphing a Polar Equation (Spiral)

Graph the equation: $r = e^{\theta/5}$

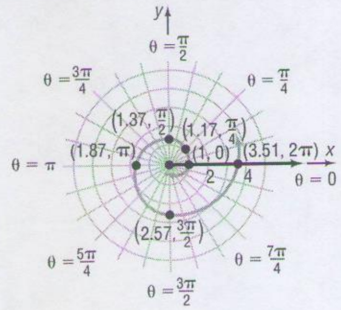
Solution The tests for symmetry with respect to the pole, the polar axis, and the line $\theta = \frac{\pi}{2}$ fail. Furthermore, there is no number θ for which $r = 0$, so the graph does not pass through the pole. Observe that r is positive for all θ, r increases as θ increases, $r \to 0$

as $\theta \to -\infty$, and $r \to \infty$ as $\theta \to \infty$. With the help of a calculator, we obtain the values in Table 6. See Figure 32.

Table 6

θ	$r = e^{\theta/5}$
$-\frac{3\pi}{2}$	0.39
$-\pi$	0.53
$-\frac{\pi}{2}$	0.73
$-\frac{\pi}{4}$	0.85
0	1
$\frac{\pi}{4}$	1.17
$\frac{\pi}{2}$	1.37
π	1.87
$\frac{3\pi}{2}$	2.57
2π	3.51

Figure 32
$r = e^{\theta/5}$

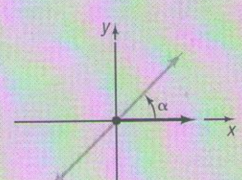
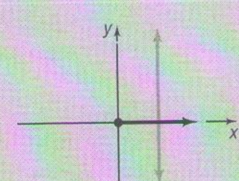
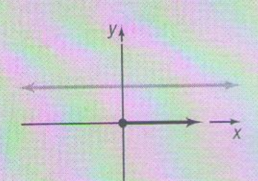
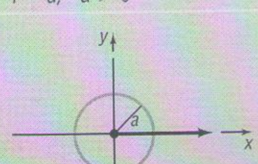
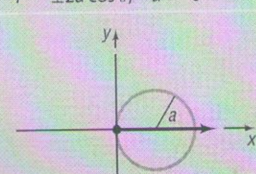
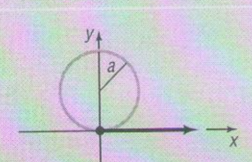


The curve in Figure 32 is called a **logarithmic spiral,** since its equation may be written as $\theta = 5 \ln r$ and it spirals infinitely both toward the pole and away from it.

Classification of Polar Equations

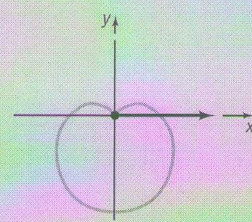
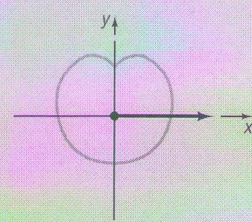
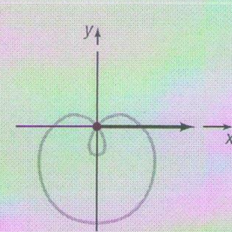
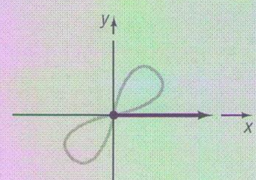
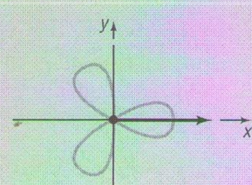
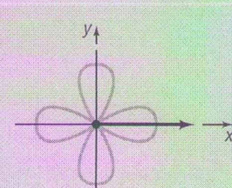
The equations of some lines and circles in polar coordinates and their corresponding equations in rectangular coordinates are given in Table 7. Also included are the names and graphs of a few of the more frequently encountered polar equations.

Table 7

Lines			
Description	Line passing through the pole making an angle α with the polar axis	Vertical line	Horizontal line
Rectangular equation	$y = (\tan \alpha)x$	$x = a$	$y = b$
Polar equation	$\theta = \alpha$	$r \cos \theta = a$	$r \sin \theta = b$
Typical graph			
Circles			
Description	Center at the pole, radius a	Passing through the pole, tangent to the line $\theta = \frac{\pi}{2}$, center on the polar axis, radius a	Passing through the pole, tangent to the polar axis, center on the line $\theta = \frac{\pi}{2}$, radius a
Rectangular equation	$x^2 + y^2 = a^2, \quad a > 0$	$x^2 + y^2 = \pm 2ax, \quad a > 0$	$x^2 + y^2 = \pm 2ay, \quad a > 0$
Polar equation	$r = a, \quad a > 0$	$r = \pm 2a \cos \theta, \quad a > 0$	$r = \pm 2a \sin \theta, \quad a > 0$
Typical graph			

(continued)

Table 7 (Continued)

Other Equations			
Name	Cardioid	Limaçon without inner loop	Limaçon with inner loop
Polar equations	$r = a \pm a \cos \theta, \quad a > 0$	$r = a \pm b \cos \theta, \quad 0 < b < a$	$r = a \pm b \cos \theta, \quad 0 < a < b$
	$r = a \pm a \sin \theta, \quad a > 0$	$r = a \pm b \sin \theta, \quad 0 < b < a$	$r = a \pm b \sin \theta, \quad 0 < a < b$
Typical graph			
Name	Lemniscate	Rose with three petals	Rose with four petals
Polar equations	$r^2 = a^2 \cos(2\theta), \quad a > 0$	$r = a \sin(3\theta), \quad a > 0$	$r = a \sin(2\theta), \quad a > 0$
	$r^2 = a^2 \sin(2\theta), \quad a > 0$	$r = a \cos(3\theta), \quad a > 0$	$r = a \cos(2\theta), \quad a > 0$
Typical graph			

Sketching Quickly

If a polar equation involves only a sine (or cosine) function, you can quickly obtain a sketch of its graph by making use of Table 7, periodicity, and a short table.

EXAMPLE 13 Sketching the Graph of a Polar Equation Quickly

Graph the equation: $r = 2 + 2 \sin \theta$

Solution You should recognize the polar equation: Its graph is a cardioid. The period of $\sin \theta$ is 2π, so form a table using $0 \le \theta \le 2\pi$, compute r, plot the points (r, θ), and sketch the graph of a cardioid as θ varies from 0 to 2π. See Table 8 and Figure 33.

Table 8

θ	$r = 2 + 2 \sin \theta$
0	$2 + 2(0) = 2$
$\frac{\pi}{2}$	$2 + 2(1) = 4$
π	$2 + 2(0) = 2$
$\frac{3\pi}{2}$	$2 + 2(-1) = 0$
2π	$2 + 2(0) = 2$

Figure 33
$r = 2 + 2 \sin \theta$