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Numerical Methods

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1- Advanced Engineering Mathematics H.K.Dass 2008

4

DETERMINANTS AND MATRICES

4.1. INTRODUCTION

In Engineering Mathematics, solution of simultaneous equations is very important. In this chapter we shall study the system of linear equations with emphasis on their solution by means of determinants.

4.2. DETERMINANT

The notation of determinants arises from the process of elimination of the unknowns of simultaneous linear equations.

Consider the two linear equations in x,

$$a_1 x + b_1 = 0 \qquad \text{... (1)}$$

$$a_2 x + b_2 = 0 \qquad \text{... (2)}$$

From (1) $\qquad x = -\dfrac{b_1}{a_1}$

Substituting the value of x in (2); we get the eliminant

$$a_2\left(-\frac{b_1}{a_1}\right) + b_2 = 0$$

or $\qquad a_1 b_2 - a_2 b_1 = 0 \qquad$ ... (3)

From (1) and (2) by supressing x, the eliminant is written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \qquad \text{... (4)}$$

when the two **rows** of a_1, b_1 and a_2, b_2 are enclosed by two vertical bars then it is called a **determinant of second order.**

$$\begin{vmatrix} a_1 \\ \\ a_2 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} b_1 \\ \\ b_2 \end{vmatrix}$$

Column 1 $\qquad$ Column 2

Row 1 $\rightarrow$ $a_1 \cdots\cdots b_1$

Row 2 $\rightarrow$ $a_2 \cdots\cdots b_2$

Each quantity a_1, b_1, a_2, b_2 is called an **element** or a constituent of the determinant.

From (3) and (4), we know that both expressions are eliminant, so we equate them.

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1 \qquad \text{or} \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$$

$a_1b_2 - a_2b_1$ is called the expansion of the determinant of $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

Example 1. *Expand the determinant* $\begin{vmatrix} 3 & 2 \\ 6 & 7 \end{vmatrix}$.

Solution. $\begin{vmatrix} \overset{+}{3} & \overset{-}{2} \\ 6 & 7 \end{vmatrix} = (3) \times (7) - (2) \times (6) = 21 - 12 = 9.$ **Ans.**

EXERCISE 4.1

Expand the following determinants :

1. $\begin{vmatrix} 4 & 6 \\ 2 & 5 \end{vmatrix}$ **Ans.** 8

2. $\begin{vmatrix} -3 & 7 \\ 2 & 4 \end{vmatrix}$ **Ans.** -26

3. $\begin{vmatrix} 8 & 5 \\ 3 & 1 \end{vmatrix}$ **Ans.** -7

4. $\begin{vmatrix} 5 & -2 \\ 4 & 3 \end{vmatrix}$ **Ans.** 23

4.3. DETERMINANT AS ELIMINANT

Consider the following three equations having three unknowns, x, y and z.

$$a_1 x + b_1 y + c_1 z = 0 \quad ...(1)$$
$$a_2 x + b_2 y + c_2 z = 0 \quad ...(2)$$
$$a_3 x + b_3 y + c_3 z = 0 \quad ...(3)$$

From (2) and (3) by cross-multiplication, we get

$$\frac{x}{b_2 c_3 - b_3 c_2} = \frac{y}{a_3 c_2 - a_2 c_3} = \frac{z}{a_2 b_3 - a_3 b_2} = k \text{ (say)}$$

$$x = (b_2 c_3 - b_3 c_2)\, k$$
$$y = (a_3 c_2 - a_2 c_3)\, k$$

and $$z = (a_2 b_3 - a_3 b_2)\, k$$

Substituting the values of x, y and z in (1), we get the eliminant

$$a_1 (b_2c_3 - b_3c_2)\, k + b_1 (a_3c_2 - a_2c_3)\, k + c_1 (a_2b_3 - a_3b_2)\, k = 0$$

or $$a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2) = 0 \quad ...(4)$$

From (1), (2) and (3) by suppressing x, y, z the remaining can be written in the determinant as

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad ...(5)$$

This is determinant of third order.

As (4) and (5) both are the eliminant of the same equations.

$$\therefore \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2) = 0$$

or $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

4.4. MINOR

The minor of an element is defined as a determinant obtained by deleting the row and column containing the element.

Thus the minors a_1, b_1 and c_1 are respectively.

$$\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} \text{ and } \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

Thus

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 (\text{minor of } a_1) - b_1 (\text{minor of } b_1) + c_1 (\text{minor of } c_1).$$

4.5. COFACTOR

Cofactor $= (-1)^{r+c}$ Minor

where r is the number of rows of the element and c is the number of columns of the element.

The cofactor of any element of jth row and ith column is

$$(-1)^{i+j} \text{ minor}$$

Thus the cofactor of $a_1 = (-1)^{1+1}(b_2c_3 - b_3c_2) = +(b_2c_3 - b_3c_2)$

The cofactor of $b_1 = (-1)^{1+2}(a_2c_3 - a_3c_2) = -(a_2c_3 - a_3c_2)$

The cofactor of $c_1 = (-1)^{1+3}(a_2b_3 - a_3b_2) = +(a_2b_3 - a_3b_2)$

The determinant $= a_1$ (cofactor of a_1) $+ a_2$ (cofactor of a_2) $+ a_3$ (cofactor of a_3).

SOLVED EXAMPLES

Example 2. *Write down the minors and cofactors of each element and also evaluate the determinant.*

$$\begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix}$$

Solution. M_{11} = Minor of element (1) $= \begin{vmatrix} 1 & \cdots 3 \cdots & -2 \\ \vdots 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix}$

$$= \begin{vmatrix} -5 & 6 \\ 5 & 2 \end{vmatrix} = (-5) \times 2 - 6 \times 5 = -10 - 30 = -40$$

Cofactor of element (1) $= A_{11} = (-1)^{1+1} M_{11} = (-1)^2(-40) = -40$

M_{12} = Minor of element (3)

$$= \begin{vmatrix} 1 & \cdots 3 \cdots & -2 \\ 4 & \vdots -5 & 6 \\ 3 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 4 & 6 \\ 3 & 2 \end{vmatrix} = 4 \times 2 - 3 \times 6 = 8 - 18 = -10$$

$\Rightarrow$ Cofactor of element $(-2) = A_{12} = (-1)^{1+2}(-10) = 10$

M_{13} = Minor of element (-2)

$$= \begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 4 & -5 \\ 3 & 5 \end{vmatrix} = 4 \times 5 - (-5) \times 3 = 20 + 15 = 35$$

$\Rightarrow$ Cofactor of element $(-2) = A_{13} = (-1)^{1+3} M_{13} = (-1)^4\, 35 = 35$

M_{21} = Minor of element (4)

$$= \begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} = 3 \times 2 - (-2) \times 5 = 6 + 10 = 16$$

$\Rightarrow$ Cofactor of element $(4) = A_{21} = (-1)^{2+1} M_{21} = (-1)^{2+1}(16) = -16$

M_{22} = Minor of element (-5)

$$= \begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = 1 \times 2 - (-2) \times 3 = 2 + 6 = 8$$

$\Rightarrow$ Cofactor of element $(-5) = A_{22} = (-1)^{2+2} M_{22} = (-1)^{2+2}(8) = 8$

M_{23} = Minor of element (6)

$$= \begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = 1 \times 5 - 3 \times 3 = 5 - 9 = -4$$

$\Rightarrow$ Cofactor of element $(6) = A_{23} = (-1)^{2+3} M_{23} = (-1)^{2+3}(-4) = 4$

M_{31} = Minor of element (3)

$$= \begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ -5 & 6 \end{vmatrix} = 3 \times 6 - (-2) \times (-5) = 18 - 10 = 8$$

$\Rightarrow$ Cofactor of element $(3) = A_{31} = (-1)^{3+1} M_{31} = (-1)^{3+1}\, 8 = 8$

M_{32} = Minor of element (5)

$$= \begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 4 & 6 \end{vmatrix} = 1 \times 6 - (-2) \times 4 = 6 + 8 = 14$$

$\Rightarrow$ Cofactor of element $(5) = A_{32} = (-1)^{3+2} M_{32} = (-1)^{3+2}\, 14 = -14$

M_{33} = Minor of element (2)

$$= \begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 4 & -5 \end{vmatrix} = 1 \times (-5) - 4 \times 3 = -5 - 12 = -17$$

Cofactor of element $(2) = A_{33} = (-1)^{3+3} M_{33} = (-1)^{3+3}(-17) = -17$.

$$\begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix} = 1 \times (\text{cofactor of } 1) + 3 \times (\text{cofactor of } 3) + (-2) \times [\text{cofactor of } (-2)].$$

$$= 1 \times (-40) + 3 \times (10) + (-2) \times (35)$$

$$= -40 + 30 - 70$$

$$= -80$$

Ans.

Example 3. *Find :*

(*i*) *Minors* (*ii*) *Cofactors of the elements of the first row of the determinant*

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix}$$

Solution.

(*i*) The minor of the element (2) is

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 2 & 7 \end{vmatrix} = (1)\times(7)-(0)\times(2) = 7-0 = 7$$

The minor of the element (3) is

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 0 \\ 6 & 7 \end{vmatrix} = (4)\times(7)-(0)\times(6) = 28-0 = 28$$

The minor of the element (5) is

$$\begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 0 \\ 6 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix} = (4)\times(2)-(1)\times(6) = 8-6 = 2$$

(*ii*) The cofactor of (2) $= (-1)^{1+1}\,(7) = +7$

The cofactor of (3) $= (-1)^{1+2}\,(28) = -28$

The cofactor of (5) $= (-1)^{1+3}\,(2) = +2$. **Ans.**

Example 4. *Expand the determinant* $\begin{vmatrix} 6 & 2 & 3 \\ 2 & 3 & 5 \\ 4 & 2 & 1 \end{vmatrix}$

Solution. $\begin{vmatrix} 6 & 2 & 3 \\ 2 & 3 & 5 \\ 4 & 2 & 1 \end{vmatrix}$ = 6 (cofactor of 6) + 2 (cofactor of 2) + 3 (cofactor of 3).

$$\begin{aligned}
&= 6\,(3\times1-5\times2)-2\,(2\times1-4\times5)+3\,(2\times2-3\times4) \\
&= 6\,(3-10)-2\,(2-20)+3\,(4-12) \\
&= 6\,(-7)-2\,(-18)+3\,(-8) \\
&= -42+36-24 \\
&= -30.
\end{aligned}$$

Ans.

Example 5. *Evaluate the determinant* $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$.

(*i*) *With the help of second row,* (*ii*) *with the help of third column.*

Solution.

(*i*) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 3\times$ (cofactor of 3) $+\,5\times$ (cofactor of 5) $+\,(-1)$ (cofactor of -1).

$$= 3 \times (-1)^{2+1} \begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} + 5 \times (-1)^{2+2} \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} + (-1) \times (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= -3 \times (0-4) + 5(2-0) + (1-0)$$

$$= 12 + 10 + 1 = 23$$ **Ans.**

(*ii*) $$\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 4 \times (\text{cofactor of } 4) + (-1)(\text{cofactor of } (-1)) + 2 \times (\text{cofactor of } 2)$$

$$= 4 \times (-1)^{1+3} \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} + (-1)(-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + 2 \times (-1)^{3+3} \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix}$$

$$= 4 \times (3-0) + (1-0) + 2(5-0)$$

$$= 12 + 1 + 10 = 23$$ **Ans.**

Example 6. *Expand the fourth order determinant* $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{vmatrix}$

Solution. Given determinant $= (0)(-1)^{1+1} \begin{vmatrix} 0 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{vmatrix} + 1(-1)^{1+2} \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix}$

$$+ 2(-1)^{1+3} \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} + 3(-1)^{1+4} \begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 0 - \begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} - 3 \begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

Now $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix} = 1(1 \times 0 - 3 \times 1) - 2(2 \times 0 - 3 \times 1) + 0(2 \times 1 - 1 \times 1)$

$$= -3 + 6 + 0 = 3$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} = 1(0 \times 0 - 3 \times 2) - 0(2 \times 0 - 3 \times 1) + 0(2 \times 2 - 0 \times 1)$$

$$= -6$$

$$\begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1(0 \times 1 - 1 \times 2) - 0(2 \times 1 - 1 \times 1) + 2(2 \times 2 - 0 \times 1)$$

$$= -2 - 0 + 8 = 6$$

Now $\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 2 & 1 & 0 \end{vmatrix} = -3 + 2(-6) - 3(6)$

$$= -3 - 12 - 18 = -33$$ **Ans.**

EXERCISE 4.2

Write the minors and co factors of each element of the following determinants and also evalutate the determinant in each case :

1. $\begin{vmatrix} -2 & 3 \\ 4 & -9 \end{vmatrix}$ $M_{11} = -9,\ M_{12} = 4,\ M_{21} = 3,\ M_{22} = -2$
$A_{11} = -9,\ A_{12} = -4,\ A_{21} = -3,\ A_{22} = -2 \quad |A| = 6$ **Ans.**

2. $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ $M_{11} = \cos\theta,\ M_{12} = \sin\theta,\ M_{21} = -\sin\theta,\ M_{22} = \cos\theta$
$A_{11} = \cos\theta,\ A_{12} = -\sin\theta,\ A_{21} = \sin\theta,\ A_{22} = \cos\theta, \quad |A| = 1$ **Ans.**

3. $\begin{vmatrix} 42 & 1 & 6 \\ 28 & 7 & 4 \\ 14 & 3 & 2 \end{vmatrix}$ $M_{11} = 2,\ M_{12} = 0,\ M_{13} = -14,\ M_{21} = -16,\ M_{22} = 0$
$M_{23} = 112,\ M_{31} = -38,\ M_{32} = 0,\ M_{33} = 266$
$A_{11} = 2,\ A_{12} = 0,\ A_{13} = -14,\ A_{21} = 16,\ A_{22} = 0$
$A_{23} = -112,\ A_{31} = -38,\ A_{32} = 0,\ A_{33} = 266, \quad |A| = 0$ **Ans.**

4. $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ $M_{11} = (ab^2 - ac^2),\ M_{12} = (ab - ac),\ M_{13} = (c - b),\ M_{21} = a^2b - bc^2$
$M_{22} = (ab - bc),\ M_{23} = (c - a),\ M_{31} = (ca^2 - cb^2),\ M_{32} = ca - bc,\ M_{33} = (b - a),$
$A_{11} = (ab^2 - ac^2),\ A_{12} = (ac - ab),\ A_{13} = (c - b),\ A_{21} = bc^2 - a^2b$
$A_{22} = (ab - bc),\ A_{23} = (a - c),\ A_{31} = (ca^2 - cb^2),\ A_{32} = (bc - ca),\ A_{33} = (b - a)$
$|A| = (a - b)(b - c)(c - a).$ **Ans.**

Expand the following determinants :

5. $\begin{vmatrix} 2 & -3 & 4 \\ 5 & 1 & -6 \\ -7 & 8 & -9 \end{vmatrix}$ **Ans.** $|A| = 5$

6. $\begin{vmatrix} 5 & 0 & 7 \\ 8 & -6 & -4 \\ 2 & 3 & 9 \end{vmatrix}$ **Ans.** $|A| = 42$

7. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ **Ans.** $|A| = abc + 2fgh - af^2 - bg^2 - ch^2$

Expand the following determinants by two methods :

(*i*) along the-third row.

(*ii*) along the-third column.

8. $\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$ **Ans.** $|A| = 40$

9. $\begin{vmatrix} 3 & -2 & 4 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{vmatrix}$ **Ans.** $|A| = -7$

10. $\begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$ **Ans.** $|A| = -37$

11. $\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix}$ **Ans.** $|A| = \dfrac{15}{2}$

12. If a, b, c are all positive and are the pth, qth, rth terms of a *G.P.* respectively; then prove that

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$$

13. $\begin{vmatrix} 3 & 2 & 5 & 7 \\ -1 & -4 & -3 & 0 \\ 6 & 4 & 2 & -1 \\ 2 & -1 & 0 & 3 \end{vmatrix}$ **Ans.** 96

4.6. RULES OF SARRUS (For third order determinants only).

After writing the determinant, repeat the first two columns as below

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{array}{ccc|cc} + & + & + & - & - \quad - \\ a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \end{array}$$

$$= (a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3) + (-c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3)$$

Example 7. *Expand the determinant*

$$\Delta = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 5 & 3 \\ 3 & 0 & 5 \end{vmatrix} \text{ by Rule of Sarrus.}$$

Solution.

$$\Delta = \begin{array}{ccc|cc} + & + & + & - & - \quad - \\ 2 & 3 & 4 & 2 & 3 \\ 1 & 5 & 3 & 1 & 5 \\ 3 & 0 & 5 & 3 & 0 \end{array}$$

$$= (2) \times (5) \times (5) + (3) \times (3) \times (3) + (4) \times (1) \times (0) - (4) \times (5) \times (3) - (2) \times (3) \times (0) - (3) \times (1) \times (5)$$

$$= 50 + 27 + 0 - 60 - 0 - 15 = 2$$ **Ans.**

EXERCISE 4.3

Expand the following determinants by Rule of Sarrus.

1. $\begin{vmatrix} 3 & 2 & -4 \\ 5 & 1 & -1 \\ -2 & 6 & 7 \end{vmatrix}$ **Ans.** -155

2. $\begin{vmatrix} 1 & 4 & 2 \\ 2 & 5 & 3 \\ 3 & 6 & 4 \end{vmatrix}$ **Ans.** 0

3. $\begin{vmatrix} 6 & 3 & 7 \\ 32 & 13 & 37 \\ 10 & 4 & 11 \end{vmatrix}$ **Ans.** 10

4. $\begin{vmatrix} 9 & 25 & 6 \\ 7 & 13 & 5 \\ 9 & 23 & 6 \end{vmatrix}$ **Ans.** 6

5. If $a + b + c = 0$, solve the equation $\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0$

Ans. $x = \pm\sqrt{(a^2+b^2+c^2-ab-bc-ca)}$, $x = 0$

4.7. PROPERTIES OF DETERMINANTS

Property (*i*) The value of a determinant remains unaltered, if the rows are interchanged into columns (or the columns into rows).

Consider the determinant.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1$$

$$= (a_1b_2c_3 - a_1b_3c_2) - (a_2b_1c_3 - a_2b_3c_1) + (a_3b_1c_2 - a_3b_2c_1)$$

$$= a_1 (b_2c_3 - b_3c_2) - a_2 (b_1c_3 - b_3c_1) + a_3 (b_1c_2 - b_2c_1)$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Proved.

Property (*ii*) If two rows (or two columns) of a determinant are interchanged, the sign of the value of the determinant changes.

Interchanging the first two rows of Δ, we get

$$\Delta' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_2 (b_1c_3 - b_3c_1) - b_2 (a_1c_3 - a_3c_1) + c_2 (a_1b_3 - a_3b_1)$$

$$= a_2b_1c_3 - a_2b_3c_1 - a_1b_2c_3 + a_3b_2c_1 + a_1b_3c_2 - a_3b_1c_2$$

$$= -[(a_1b_2c_3 - a_1b_3c_2) - (a_2b_1c_3 - a_3b_1c_2) + (a_2b_3c_1 - a_3b_2c_1)]$$

$$= -[(a_1 (b_2c_3 - b_3c_2) - b_1 (a_2c_3 - a_3c_2) + c_1 (a_2b_3 - a_3b_2)]$$

$$= -\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\Delta$$

Proved.

Property (*iii*) If two rows (or columns) of a determinant are identical, the value of the determinant is zero.

Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$, so that the first two rows are identical.

By interchanging the first two rows, we get the same determinant Δ.

By property (*ii*), on interchanging the rows, the sign of the determinant changes.

or $\Delta = -\Delta$ or $2\Delta = 0$ or $\Delta = 0$ **Proved.**

Property (*iv*) If the elements of any row (or column) of a determinant be each multiplied by the same number, the determinant is multiplied by that number.

$$\Delta' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= ka_1(b_2c_3 - b_3c_2) - kb_1(a_2c_3 - a_3c_2) + kc_1(a_2b_3 - a_3b_2)$$
$$= k[a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)]$$
$$= k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\,\Delta.$$

Example 8. *Prove that*

$$\begin{vmatrix} a^2 & a & bc \\ b^2 & b & ca \\ c^2 & c & ab \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

Solution. $\begin{vmatrix} a^2 & a & bc \\ b^2 & b & ca \\ c^2 & c & ab \end{vmatrix}$

By multiplying R_1, R_2, R_3 by a, b and c respectively we get

$$= \frac{1}{abc}\begin{vmatrix} a^3 & a^2 & abc \\ b^3 & b^2 & abc \\ c^3 & c^2 & abc \end{vmatrix} = \frac{abc}{abc}\begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$ By changing rows into columns

Proved

Example 9. *Without expanding and or evaluating, show that*

$$\begin{vmatrix} a^2 & a & 1 & bcd \\ b^2 & b & 1 & cda \\ c^2 & c & 1 & dab \\ d^2 & d & 1 & abc \end{vmatrix} = \begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$$

Solution

$$\begin{vmatrix} a^2 & a & 1 & bcd \\ b^2 & b & 1 & cda \\ c^2 & c & 1 & dab \\ d^2 & d & 1 & abc \end{vmatrix} = \frac{1}{abcd}\begin{vmatrix} a^3 & a^2 & a & abcd \\ b^3 & b^2 & b & abcd \\ c^3 & c^2 & c & abcd \\ d^3 & d^2 & d & abcd \end{vmatrix} \quad \begin{matrix} R_1 \to aR_1 \\ R_2 \to bR_2 \\ R_3 \to cR_3 \\ R_4 \to dR_4 \end{matrix}$$

$$= \frac{abcd}{abcd}\begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix} C_4 \to \frac{1}{abcd} C_4 \quad = \begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$$

Proved

Example 10. *Prove that* $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ *(Try yourself)*

Property (v) The value of the determinant remains unaltered if to the elements of one row (or column) be added any constant multiple of the corresponding elements of any other row (or column) respectively.

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

On multiplying the second column by l and the third column by m and adding to the first column we get

$$\Delta' = \begin{vmatrix} a_1 + lb_1 + mc_1 & b_1 & c_1 \\ a_2 + lb_2 + mc_2 & b_2 & c_2 \\ a_3 + lb_3 + mc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + l\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + m\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

$= \Delta + 0 + 0$ (Since columns are identical)

$= \Delta$

Proved

Example 11. *Evaluate, using the properties of determinant* $\begin{vmatrix} 9 & 9 & 12 \\ 1 & 3 & -4 \\ 1 & 9 & 12 \end{vmatrix}$

Solution. Let $$\Delta = \begin{vmatrix} 9 & 9 & 12 \\ 1 & 3 & -4 \\ 1 & 9 & 12 \end{vmatrix}$$

Applying : $R_1 \to R_1 + 3R_2$ and $R_3 \to R_3 + 3R_2$, we get

$$\Delta = \begin{vmatrix} 12 & 18 & 0 \\ 1 & 3 & -4 \\ 4 & 18 & 0 \end{vmatrix} = 6 \times 2 \begin{vmatrix} 2 & 3 & 0 \\ 1 & 3 & -4 \\ 2 & 9 & 0 \end{vmatrix}$$

Expand by C_3 $$\Delta = 6 \times 2 \times 4 \begin{vmatrix} 2 & 3 \\ 2 & 9 \end{vmatrix}$$

$$= 48\,(2 \times 9 - 2 \times 3) = 48 \times 12 = 576.$$

Ans.

Example 12. *Without expanding evaluate the determinant* $\Delta = \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$

Solution. Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} 46 & 21 & 219 \\ 42 & 27 & 198 \\ 38 & 17 & 181 \end{vmatrix}$$

Applying $C_1 \to C_1 - 2C_2$ and $C_3 \to C_3 - 10\,C_2$, we get

$$\Delta = \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 + 3R_3$

$$\Delta = \begin{vmatrix} 0 & 4 & -2 \\ 0 & 78 & -39 \\ 4 & 17 & 11 \end{vmatrix} = 2(39)\begin{vmatrix} 0 & 2 & -1 \\ 0 & 2 & -1 \\ 4 & 17 & 11 \end{vmatrix}$$ [Taking 2 common from R_1 and 39 common from R_2]

$= 78 \times 0 = 0$ (Since R_1 and R_2 are identical) **Ans.**

Example 13. *Show that* $\Delta = \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix} = 0$

Solution. Let $\Delta = \begin{vmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} 0 & c-a & a-b \\ 0 & a-b & b-c \\ 0 & b-c & c-a \end{vmatrix} = 0$$ [$\because$ C_1 consists of all zeros.]

Example 14. *Without expanding, evaluate the determinant* $\begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$.

Solution. Let $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin(\alpha+\delta) \\ \sin\beta & \cos\beta & \sin(\beta+\delta) \\ \sin\gamma & \cos\gamma & \sin(\gamma+\delta) \end{vmatrix}$

$$\Rightarrow \quad \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \sin\alpha\cos\delta + \cos\alpha\sin\delta \\ \sin\beta & \cos\beta & \sin\beta\cos\delta + \cos\beta\sin\delta \\ \sin\gamma & \cos\gamma & \sin\gamma\cos\delta + \cos\gamma\sin\delta \end{vmatrix}$$

[$\because$ $\sin(A+B) = \sin A\cos B + \cos A\sin B$]

$$\Rightarrow \qquad \Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix} \quad [\text{Applying } C_3 \to C_3 - \cos\delta.C_1 - \sin\delta.C_2]$$

$$\Rightarrow \qquad \Delta = 0 \qquad [\because C_3 \text{ consists of all zeros}]$$ **Ans.**

Example 15. *Solve the determinenial equation* $\begin{vmatrix} 2x-1 & x+7 & x+4 \\ x & 6 & 2 \\ x-1 & x+1 & 3 \end{vmatrix} = 0$

Solution. Given equation $\begin{vmatrix} 2x-1 & x+7 & x+4 \\ x & 6 & 2 \\ x-1 & x+1 & 3 \end{vmatrix} = 0$

By applying $R_1 \to R_1 - (R_2 + R_3)$, we get $\begin{vmatrix} 0 & 0 & x-1 \\ x & 6 & 2 \\ x-1 & x+1 & 3 \end{vmatrix} = 0$

On expanding by first row, we get

$(x-1)(x^2 + x - 6x + 6) = 0 \Rightarrow (x-1)(x-2)(x-3) = 0 \Rightarrow x = 1, 2, 3$ **Ans.**

Example 16. *Using the properties of determinants, show that*

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix} = x^3.$$

Solution. Let $\Delta = \begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}$

Operate : $R_2 \to R_2 - 2R_1$; $R_3 \to R_3 - 3R_1$

$$\Delta = \begin{vmatrix} x+y & x & x \\ 3x+2y & 2x & 0 \\ 7x+5y & 5x & 0 \end{vmatrix} \quad \text{Expand by } C_3 \quad \Delta = x\begin{vmatrix} 3x+2y & 2x \\ 7x+5y & 5x \end{vmatrix}$$

$$= x[5x(3x+2y) - 2x(7x+5y)]$$
$$= x[15x^2 + 10xy - (14x^2 + 10xy)] = x^3.$$ **Proved.**

Example 17. *Using the properties of determinants, evaluate the following :*

$$\begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & cb^2 & 0 \end{vmatrix}$$

Solution. Let $\Delta = \begin{vmatrix} 0 & ab^2 & ac^2 \\ a^2b & 0 & bc^2 \\ a^2c & cb^2 & 0 \end{vmatrix}$

Take a^2, b^2 and c^2 common from C_1, C_2 and C_3 respectively, $\Delta = a^2b^2c^2\begin{vmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{vmatrix}$

Operate : $C_2 \to C_2 - C_3$, $\Delta = a^2b^2c^2 \begin{vmatrix} 0 & 0 & a \\ b & -b & b \\ c & c & 0 \end{vmatrix}$

Expand by R_1, $\Delta = a^2b^2c^2 . a \begin{vmatrix} b & -b \\ c & c \end{vmatrix} = a^3b^2c^2(bc + bc) = 2a^3b^3c^3$. **Ans.**

Example 18. *Using properties of determinants, prove that*

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x).$$

Solution. Let $\Delta = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

Operate : $C_1 \to C_1 - C_2$; $C_2 \to C_2 - C_3$, $\Delta = xyz \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix}$

On expanding by R_1, $\Delta = xyz \begin{vmatrix} x-y & y-z \\ x^2-y^2 & y^2-z^2 \end{vmatrix} = xyz(x-y)(y-z) \begin{vmatrix} 1 & 1 \\ x+y & y+z \end{vmatrix}$

$= xyz(x-y)(y-z)(z-x)$. **Proved.**

Example 19. *Using the properties of determinants, show that*

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z).$$

Solution. Let $\Delta = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

Operate : $R_1 \to R_1 - R_2$, $\Delta = \begin{vmatrix} a & -a & 0 \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$

Operate : $C_2 \to C_2 + C_1$, $\Delta = \begin{vmatrix} a & 0 & 0 \\ x & a+y+x & z \\ x & y+x & a+z \end{vmatrix}$

On expanding by R_1 $\Delta = a \begin{vmatrix} a+y+x & z \\ y+x & a+z \end{vmatrix} = a[(a+y+x)(a+z) - (y+x)z]$

$= a[a^2 + az + (y+x)a + (y+x)z - (y+x)z]$

$= a^2(a+x+y+z)$. **Proved.**

Example 20. *If ω is the one of the imaginary cube roots of unity, find the value of the determinant*

$$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$$

Solution. The given determinant $= \begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$

By $R_1 \to R_1 + R_2 + R_3$, we get

$$= \begin{vmatrix} 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} \quad [\because\ 1+\omega+\omega^2=0]$$

$= 0$ (Since each entry in R_1 is zero) **Ans.**

Example 21. *Without expanding the determinant, show that $(a + b + c)$ is a factor of* $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

Solution. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

Operate : $C_1 \to C_1 + C_2 + C_3$, $\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$

$\Rightarrow$ $(a + b + c)$ is a factor of Δ. **Proved.**

Example 22. *Using properties of determinants, prove that :*

$$\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = 16(3x+4)$$

Solution. Let $\Delta = \begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$

Operate : $C_1 \to C_1 + C_2 + C_3$, $\Delta = \begin{vmatrix} 3x+4 & x & x \\ 3x+4 & x+4 & x \\ 3x+4 & x & x+4 \end{vmatrix}$

$$= (3x+4)\begin{vmatrix} 1 & x & x \\ 1 & x+4 & x \\ 1 & x & x+4 \end{vmatrix} = (3x+4)\begin{vmatrix} 1 & x & x \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}\begin{matrix} \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} = (3x+4)\begin{vmatrix} 4 & 0 \\ 0 & 4 \end{vmatrix}$$

$= 16(3x + 4)$ **Proved.**

Example 23. *Without expanding the determinant, prove that* $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0.$

Solution. Let
$$\Delta = \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$$

Operate : $C_3 \to C_3 + C_2$,
$$\Delta = \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & a+b+c \\ 1 & c & a+b+c \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$= 0$ ($\because$ C_1 and C_3 are identical). **Proved.**

Example 24. *Without expanding the determinant, prove that* $\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix} = 0$

Solution. Let
$$\Delta = \begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$$

Multiply R_1 by a, R_2 by b and R_3 by c.
$$\Delta = \frac{1}{abc}\begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & abc \\ 1 & c^3 & abc \end{vmatrix} = \frac{1}{abc}.abc\begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix} = 1 \times 0 = 0.$$

(Since C_1 and C_3 are identical) **Proved.**

Example 25. *Evaluate* $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

Solution. Let Δ be the given determinant. Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we get,
$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}$$
[Taking out $(b-a)$ common from R_2 and $(c-a)$ from R_3]

$$= (b-a)(c-a)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}$$
[Applying $R_3 \to R_3 - R_2$]

$$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 0 & c-b \end{vmatrix} \qquad \text{[Expanding along } C_1\text{]}$$

$$= (b-a)(c-a)(c-b).$$

Ans.

Example 26. *Using properties of determinants, prove that :*

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

Solution. Let $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$.

Operate : $R_1 \to R_1 - R_2$; $R_2 \to R_2 - R_3$, $\Delta = \begin{vmatrix} 0 & a-b & a^3-b^3 \\ 0 & b-c & b^3-c^3 \\ 1 & c & c^3 \end{vmatrix} = 1.\begin{vmatrix} a-b & a^3-b^3 \\ b-c & b^3-c^3 \end{vmatrix}$

(Expanding by C_1)

$$= (a-b)(b-c)\begin{vmatrix} 1 & a^2+ab+b^2 \\ 1 & b^2+bc+c^2 \end{vmatrix}$$

Operate : $R_1 \to R_1 - R_2$, $\Delta = (a-b)(b-c)\begin{vmatrix} 0 & (a^2-c^2)+(ab-bc) \\ 1 & b^2+bc+c^2 \end{vmatrix}$

$$= (a-b).(b-c).(-1)[(a^2-c^2)+b(a-c)]$$

$$= (a-b).(b-c)(c-a)(a+b+c).$$

Proved.

Example 27. *Evaluate* $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Solution. By $R_1 \to R_1 + R_2 + R_3$, we get $\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix} \begin{matrix} C_2 - C_1 \\ C_3 - C_1 \end{matrix}$$

On expanding by first row $= (a+b+c)(a+b+c)^2 = (a+b+c)^3$.

Ans.

Example 28. *Show, without expanding* $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Solution. By $C_1 - C_2, C_2 - C_3$, we get $= \begin{vmatrix} 0 & 0 & 1 \\ x-y & y-z & z \\ x^2-y^2 & y^2-z^2 & z^2 \end{vmatrix} = \begin{vmatrix} x-y & y-z \\ x^2-y^2 & y^2-z^2 \end{vmatrix}$

On expanding by first row, we get

$= (x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x+y & y+z \end{vmatrix} = (x-y)(y-z)(y+z-x-y) = (x-y)(y-z)(z-x)$. **Ans.**

Example 29. *Prove that* $\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix} = (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$

Solution. Let $\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \beta+\gamma & \gamma+\alpha & \alpha+\beta \end{vmatrix}$.

$$\Delta = \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ \alpha+\beta+\gamma & \alpha+\beta+\gamma & \alpha+\beta+\gamma \end{vmatrix} \quad \text{Applying } R_3 \to R_1 + R_3$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \\ 1 & 1 & 1 \end{vmatrix} \quad [\text{Taking out } (\alpha+\beta+\gamma) \text{ common from } R_3]$$

$$= (\alpha+\beta+\gamma)\begin{vmatrix} \alpha & \beta-\alpha & \gamma-\alpha \\ \alpha^2 & \beta^2-\alpha^2 & \gamma^2-\alpha^2 \\ 1 & 0 & 0 \end{vmatrix} \quad \text{Applying } C_2 \to C_2 - C_1,\ C_3 \to C_3 - C_1$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & 1 & 1 \\ \alpha^2 & \beta+\alpha & \gamma+\alpha \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha).1\begin{vmatrix} 1 & 1 \\ \beta+\alpha & \gamma+\alpha \end{vmatrix} \quad [\text{Expanding along } R_3]$$

$$= (\alpha+\beta+\gamma)(\beta-\alpha)(\gamma-\alpha)(\gamma+\alpha-\beta-\alpha)$$

$$= (\alpha+\beta+\gamma)(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)$$ **Proved.**

Example 30. *Prove that* $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4\,a^2b^2c^2$

Solution. Let $\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$.

Taking a, b, c common from R_1, R_2 and R_3 respectively, we get $\Delta = abc\begin{vmatrix} -a & a & a \\ b & -b & b \\ c & c & -c \end{vmatrix}$

$$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \quad \text{[Taking } a, b, c \text{ common from } C_1, C_2 \text{ and } C_3 \text{ respectively]}$$

$$= a^2b^2c^2\begin{vmatrix} -1 & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{vmatrix} \quad [\text{Applying } C_2 \to C_2 + C_1,\ C_3 \to C_3 + C_1]$$

$$= a^2b^2c^2(-1)\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix} \quad [\text{Expanding along } R_1]$$

$$= a^2b^2c^2(-1)(0-4) = 4a^2b^2c^2$$

Proved.

Example 31. *Show that* $\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca)$

Solution. Let
$$\Delta = \begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we get $\Delta = \begin{vmatrix} a+b+c & -a+b & -a+c \\ a+b+c & 3b & -b+c \\ a+b+c & -c+b & 3c \end{vmatrix}$

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 1 & 3b & -b+c \\ 1 & -c+b & 3c \end{vmatrix} \quad [\text{Taking } (a+b+c) \text{ common from } C_1]$$

$$= (a+b+c)\begin{vmatrix} 1 & -a+b & -a+c \\ 0 & 2b+a & -b+a \\ 0 & -c+a & 2c+a \end{vmatrix} \quad [\text{Applying } R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1]$$

$$= (a+b+c)\begin{vmatrix} 2b+a & -b+a \\ -c+a & 2c+a \end{vmatrix} \quad [\text{Expanding along } C_1]$$

$$= (a+b+c)[(2b+a)(2c+a) - (-b+a)(-c+a)]$$
$$= (a+b+c)\{(4bc+2ab+2ca+a^2-(bc-ab-ac+a^2)\}$$
$$= (a+b+c)(3bc+3ab+3ca)$$
$$= 3(a+b+c)(ab+bc+ca)$$

Proved.

Property (*vi*) If each element of a row (or column) of a determinant consists of the algebraic sum of n terms, the determinant can be expressed as the sum of n determinants,

Let
$$\Delta = \begin{vmatrix} a_1+p_1+q_1 & b_1 & c_1 \\ a_2+p_2+q_2 & b_2 & c_2 \\ a_3+p_3+q_3 & b_3 & c_3 \end{vmatrix}.$$

$$=(a_1+p_1+q_1)(b_2c_3-b_3c_2)-(a_2+p_2+q_2)(b_1c_3-b_3c_1)+(a_3+p_3+q_3)(b_1c_2-b_2c_1)$$

$$\begin{aligned}&= a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-b_3c_1)+a_3(b_1c_2-b_2c_1)\\ &\quad + p_1(b_2c_3-b_3c_2)-p_2(b_1c_3-b_3c_1)+p_3(b_1c_2-b_2c_1)\\ &\quad + q_1(b_2c_3-b_3c_2)-q_2(b_1c_3-b_3c_1)+q_3(b_1c_2-b_2c_1)\end{aligned}$$

$$=\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}+\begin{vmatrix} p_1 & b_1 & c_1\\ p_2 & b_2 & c_2\\ p_3 & b_3 & c_3\end{vmatrix}+\begin{vmatrix} q_1 & b_1 & c_1\\ q_2 & b_2 & c_2\\ q_3 & b_3 & c_3\end{vmatrix}$$ **Proved.**

Example 32. *If* $\begin{vmatrix} a & a^2 & a^3-1\\ b & b^2 & b^3-1\\ c & c^2 & c^3-1\end{vmatrix}=0$, *prove that* $abc=1$.

Solution. $\begin{vmatrix} a & a^2 & a^3-1\\ b & b^2 & b^3-1\\ c & c^2 & c^3-1\end{vmatrix}=0 \Rightarrow \begin{vmatrix} a & a^2 & a^3\\ b & b^2 & b^3\\ c & c^2 & c^3\end{vmatrix}+\begin{vmatrix} a & a^2 & -1\\ b & b^2 & -1\\ c & c^2 & -1\end{vmatrix}=0$

$$\Rightarrow \quad abc\begin{vmatrix} 1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}-\begin{vmatrix} a & a^2 & 1\\ b & b^2 & 1\\ c & c^2 & 1\end{vmatrix}=0$$

(Taking out common a, b, c from R_1, R_2 and R_3 from 1st determinant)

$$\Rightarrow \quad abc\begin{vmatrix} 1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}+\begin{vmatrix} a & 1 & a^2\\ b & 1 & b^2\\ c & 1 & c^2\end{vmatrix}=0 \qquad \text{(Interchanging } C_2 \text{ and } C_3)$$

$$\Rightarrow \quad abc\begin{vmatrix} 1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}-\begin{vmatrix} 1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}=0 \qquad \text{(Interchanging } C_1 \text{ and } C_2)$$

$$\Rightarrow \quad (abc-1)\begin{vmatrix} 1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}=0$$

Example 33. *Show that* $\begin{vmatrix} b+c & c+a & a+b\\ q+r & r+p & p+q\\ y+z & z+x & x+y\end{vmatrix}=2\begin{vmatrix} a & b & c\\ p & q & r\\ x & y & z\end{vmatrix}$

Solution. The above determinant can be expressed as the sum of 8 determinants as given below:

$$\begin{vmatrix} b+c & c+a & a+b\\ q+r & r+p & p+q\\ y+z & z+x & x+y\end{vmatrix}=\begin{vmatrix} b & c & a\\ q & r & p\\ y & z & x\end{vmatrix}+\begin{vmatrix} b & a & a\\ q & p & p\\ y & x & x\end{vmatrix}+\begin{vmatrix} b & c & b\\ q & r & q\\ y & z & y\end{vmatrix}+\begin{vmatrix} b & a & b\\ q & p & q\\ y & x & y\end{vmatrix}$$

$$+\begin{vmatrix} c & c & a \\ r & r & p \\ z & z & x \end{vmatrix} + \begin{vmatrix} c & a & a \\ r & p & p \\ z & x & x \end{vmatrix} + \begin{vmatrix} c & c & b \\ r & r & q \\ z & z & y \end{vmatrix} + \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix}$$

$$= \begin{vmatrix} b & c & a \\ q & r & p \\ y & z & x \end{vmatrix} + 0 + 0 + 0 + 0 + 0 + 0 + \begin{vmatrix} c & a & b \\ r & p & q \\ z & x & y \end{vmatrix}$$

$$= (-1)^2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} + (-1)^2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

Proved

Example 34. *Prove that* $\begin{vmatrix} 2\alpha & \alpha + \beta & \alpha + \gamma \\ \beta + \alpha & 2\beta & \beta + \gamma \\ \gamma + \alpha & \gamma + \beta & 2\gamma \end{vmatrix} = 0$

Proved.

Solution. Given determinant $= \begin{vmatrix} \alpha + \alpha & \alpha + \beta & \alpha + \gamma \\ \beta + \alpha & \beta + \beta & \beta + \gamma \\ \gamma + \alpha & \gamma + \beta & \gamma + \gamma \end{vmatrix}$

The above determinant can be expressed as the sum of 8 determinants.

Each of the 8 determinants has either two identical columns or identical rows.

$\therefore$ Each of the resulting determinant is zero. Hence the result.

Proved.

Example 35. *Prove that* $\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = (x-\alpha)(x-\beta)(x-\gamma)$

Solution. $\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = \begin{vmatrix} x-\alpha & l & m & 1 \\ 0 & x & n & 1 \\ 0 & \beta & x & 1 \\ 0 & \beta & \gamma & 1 \end{vmatrix} \quad (C_1 \to C_1 - \alpha C_4)$

[On expanding by first column]

We get $= (x-\alpha) \begin{vmatrix} x & n & 1 \\ \beta & x & 1 \\ \beta & \gamma & 1 \end{vmatrix} = (x-\alpha) \begin{vmatrix} x-\beta & n & 1 \\ 0 & x & 1 \\ 0 & \gamma & 1 \end{vmatrix} \quad (C_1 \to C_1 - \beta C_3)$

$= (x-\alpha)(x-\beta)(x-\gamma)$ [On expanding by first column] **Proved.**

Example 36. *Show that* $x = -(a+b+c)$ *is one root of the equation:*

$\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$ *and solve the equation completely.*

Solution. By $C_1 \to C_1 + C_2 + C_3$, we get $\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+c & a \\ x+a+b+c & a & x+b \end{vmatrix} = 0$

$$\Rightarrow \qquad (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+c & a \\ 1 & a & x+b \end{vmatrix} = 0$$

$$\Rightarrow \qquad (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x-b+c & a-c \\ 0 & a-b & x+b-c \end{vmatrix} = 0, \; R_2 \to R_2 - R_1; \; R_3 \to R_3 - R_1$$

On expanding by first column, we get

$$(x+a+b+c)\,[(x-b+c)(x+b-c)-(a-b)(a-c)] = 0$$

$$\Rightarrow \qquad (x+a+b+c)\,[x^2-(b-c)^2-(a^2-ac-ab+bc)] = 0$$

$$\Rightarrow \qquad (x+a+b+c)\,(x^2-b^2-c^2+2bc-a^2+ac+ab-bc] = 0$$

$$\Rightarrow \qquad (x+a+b+c)\,(x^2-a^2-b^2-c^2+ab+bc+ca) = 0$$

Either $x+a+b+c=0 \Rightarrow \quad x=-(a+b+c)$

or $\qquad x^2-a^2-b^2-c^2+ab+bc+ca=0$

$\Rightarrow \qquad x = \pm\sqrt{a^2+b^2+c^2-ab-bc-ca}$ **Ans.**

Example 37. *Find the value of* $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$

Solution. By C_1-C_3, C_2-C_3, we get $\begin{vmatrix} (b+c)^2-a^2 & a^2-a^2 & a^2 \\ b^2-b^2 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$

$$= \begin{vmatrix} (a+b+c)(b+c-a) & 0 & a^2 \\ 0 & (a+b+c)(c+a-b) & b^2 \\ (a+b+c)(c-a-b) & (a+b+c)(c-a-b) & (a+b)^2 \end{vmatrix}$$

On taking out $(a+b+c)$ as common from 1st and 2nd column, we get

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} -a+b+c & 0 & a^2 \\ 0 & a-b+c & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \; R_3 \to R_3 - (R_1+R_2)$$

$$= -2(a+b+c)^2\begin{vmatrix} -a+b+c & 0 & a^2 \\ 0 & a-b+c & b^2 \\ b & a & -ab \end{vmatrix}$$

On expanding by first row, we get

$$= -2(a+b+c)^2\,[(-a+b+c)\{-ab(a-b+c)-ab^2\} + a^2\{0-b(a-b+c)\}]$$

$$= -2(a+b+c)^2\,[(-a+b+c)(-a^2b-abc)-a^2b(a-b+c)]$$

$$= -2ab(a+b+c)^2\,[(-a+b+c)(-a-c)-a(a-b+c)]$$

$$= -2ab(a+b+c)^2\,(a^2+ac-ab-bc-ac-c^2-a^2+ab-ac]$$

$$= -2ab(a+b+c)^2\,(-bc-ac-c^2)$$

$$= 2abc(a+b+c)^2(b+a+c)$$

$$= 2abc(a+b+c)^3.$$

Ans.

Example 38. *Using properties of determinants, solve for x :* $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

Solution. Given that $\begin{vmatrix} a+x & a-x & a-x \\ a-x & a+x & a-x \\ a-x & a-x & a+x \end{vmatrix} = 0$

Applying $C_1 \to C_1 + C_2 + C_3$ $\begin{vmatrix} 3a-x & a-x & a-x \\ 3a-x & a+x & a-x \\ 3a-x & a-x & a+x \end{vmatrix} = 0$

$$\Rightarrow (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 1 & a+x & a-x \\ 1 & a-x & a+x \end{vmatrix} = 0$$

Now, $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, $\Rightarrow (3a-x)\begin{vmatrix} 1 & a-x & a-x \\ 0 & 2x & 0 \\ 0 & 0 & 2x \end{vmatrix} = 0$

Expanding by C_1, we get $(3a-x)(4x^2-0) = 0$

$\Rightarrow 4x^2(3a-x) = 0$ $\Rightarrow$ If $4x^2 = 0$, then $x = 0$

$\Rightarrow$ If $3a - x = 0$, then $x = 3a$

Hence, $x = 0$ or $3a$ **Ans.**

Example 39. *Using properties of determinants, prove the following*

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)$$

Solution. Let $\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$$\Delta = abc\begin{vmatrix} \frac{1+a}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & \frac{1+b}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & \frac{1+c}{c} \end{vmatrix} \Rightarrow \Delta = abc\begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

Operate : $R_1 \to R_1 + R_2 + R_3$, $\Delta = abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$

Taking $\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$ common from R_1, we get

$$\Delta = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

Operate : $C_2 \to C_2 - C_1$; $C_3 \to C_3 - C_1$, $\Delta = abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$

$= abc\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+1\right)$ (On expanding by R_1) **Proved.**

Example 40. *Prove that :* $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ac \\ c & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ac).$

Solution. Let $\Delta = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ac \\ c & c^2 & ab \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} = \frac{1}{abc}.abc\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$

Operate : $R_1 \to R_1 - R_2$; $R_2 \to R_2 - R_3$, $\Delta = \begin{vmatrix} a^2-b^2 & a^3-b^3 & 0 \\ b^2-c^2 & b^3-c^3 & 0 \\ c^2 & c^3 & 1 \end{vmatrix}$

$$= (a-b)(b-c)\begin{vmatrix} a+b & a^2+ab+b^2 & 0 \\ b+c & b^2+bc+c^2 & 0 \\ c^2 & c^3 & 1 \end{vmatrix}$$

Expand by C_3 $\Delta = (a-b)(b-c).1\begin{vmatrix} a+b & a^2+ab+b^2 \\ b+c & b^2+bc+c^2 \end{vmatrix}$

Operate : $R_2 \to R_2 - R_1$ $\Delta = (a-b)(b-c)\begin{vmatrix} a+b & a^2+ab+b^2 \\ c-a & b(c-a)+(c^2-a^2) \end{vmatrix}$

$$= (a-b)(b-c)(c-a)\begin{vmatrix} a+b & a^2+ab+b^2 \\ 1 & b+c+a \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)[(a+b)(a+b+c) - 1\,.\,(a^2+ab+b^2)]$$
$$= (a-b)(b-c)(c-a)(ab+bc+ac).$$ **Proved.**

Example 41. *Evaluate* $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Solution. By $R_1+R_2+R_3$ we get $\begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix} \begin{matrix} C_2-C_1 \\ C_3-C_1 \end{matrix}$$

$$= (a+b+c)(a+b+c)^2 = (a+b+c)^3$$ [On expanding by first row] **Ans.**

EXERCISE 4.4

Expand the following determinants, using properties of the determinants :

1. $\begin{vmatrix} 1 & 3 & 7 \\ 4 & 9 & 1 \\ 2 & 7 & 6 \end{vmatrix}$ **Ans.** 51.

2. Prove that $\begin{vmatrix} x & a & a \\ a & x & a \\ a & a & x \end{vmatrix} = (x+2a)(x-a)^2.$

3. Solve the equation $\begin{vmatrix} x^3-a^3 & x^2 & x \\ b^3-a^3 & b^2 & b \\ c^3-a^3 & c^2 & c \end{vmatrix} = 0, b \neq c, bc \neq 0$ **Ans.** $x = \dfrac{a^3}{bc}, x=b, x=c$

4. Show that zero is one of the roots of the equation $\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x-c & 0 \end{vmatrix} = 0$

5. Without expanding the determinant, prove that $\begin{vmatrix} \frac{1}{a} & a & bc \\ \frac{1}{b} & b & ca \\ \frac{1}{c} & c & ab \end{vmatrix} = 0$

6. Without expanding the determinant, prove that : $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix} = 0.$

7. Using properties of determinant prove that : $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

8. $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$

9. $\begin{vmatrix} 1 & x+y & x^2+y^2 \\ 1 & y+z & y^2+z^2 \\ 1 & z+x & z^2+x^2 \end{vmatrix} = (x-y)(y-z)(z-x).$

10. $\begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} = 0$

11. $\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0.$

12. $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix} = a^3.$

13. $\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \beta\gamma & \gamma\alpha & \alpha\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta).$

14. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

15. $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

16. $\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a).$

4.8. FACTOR THEOREM

If the elements of a determinant are polynomials in a variable x and if the substitution $x = a$ makes two rows (or columns) identical, then $(x - a)$ is a factor of the determinant.

When two rows are identical, the value of the determinant is zero. The expansion of a determinant being polynomial in x vanishes on putting $x = a$, then $x - a$ is its factor by the Remainder theorem.

Example 42. *Show that* $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

Solution. If we put $x = y$, $y = z$, $z = x$ then in each case two columns become identical and the determinant vanishes.

$\therefore$ $(x-y)$, $(y-z)$, $(z-x)$ are the factors.

Since the determinant is of third degree, the other factor can be numerical only k (say).

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = k(x-y)(y-z)(z-x) \qquad \text{... (1)}$$

This leading term (product of the elements of the diagonal elements) in the given determinant is yz^2 and in the expansion

$k(x-y)(y-z)(z-x)$ we get kyz^2

Equating the coefficient of yz^2 on both sides of (1), we have

$$k = 1$$

Hence the expansion $= (x-y)(y-z)(z-x)$. **Proved.**

Example 43. *Factorize* $\Delta = \begin{vmatrix} l & l & l \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$

Solution. Putting $a = b$, $C_1 = C_2$ and hence $\Delta = 0$.

$\therefore$ $a - b$ is a factor of Δ. Similarly $b - c$, $c - a$ are also factors of Δ.

$\therefore$ $(a - b)(b - c)(c - a)$ is a third degree factor of Δ which itself is of the fifth degree as is judged from the leading term b^2c^3.

$\therefore$ The remaining factor must be of the second degree. As Δ is symmetrical in a, b, c the remaining factor must, therefore, be of the form $k(a^2 + b^2 + c^2) + l(ab + bc + ca)$

$\therefore$ $$\Delta = (a - b)(b - c)(c - a)\{k(a^2 + b^2 + c^2) + l(ab + bc + ca)\}$$

If $k \neq 0$, we shall get terms like a^4b, b^4c etc. which do not occur in Δ. Hence, k must be zero.

$\therefore$ $$\Delta = (a - b)(b - c)(c - a)\{0 + l(ab + bc + ca)\}$$

or $$\Delta = l(a - b)(b - c)(c - a)(ab + bc + ca)$$

The leading term in $\Delta = b^2c^3$. The corresponding term on R.H.S $= l\,b^2c^3$

$\therefore$ $$l = 1$$

Hence, $$\Delta = (a - b)(b - c)(c - a)(ab + bc + ca).$$ **Ans.**

Example 44. Show that $\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x).$

Solution. $$\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = xyz\,(x - y)(y - z)(z - x)$$ (see example 43).

Proved.

Example 45. Show that $\begin{vmatrix} x^3 & x^2 & x & 1 \\ \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \end{vmatrix} = (x-\alpha)(x-\beta)(x-\gamma)(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)$

Solution. If we put $x = \alpha$; $x = \beta$; $x = \gamma$; $\alpha = \beta$, $\beta = \gamma$; $\gamma = \alpha$ then two rows become identical and the determinant vanishes.

$\therefore$ $(x - \alpha)$; $(x - \beta)$; $(x - \gamma)$; $(\alpha - \beta)$; $(\beta - \gamma)$; $(\alpha - \gamma)$ are the factors.

Since the determinant is of six degree the other factor can be numerical only say k.

$$\begin{vmatrix} x^3 & x^2 & x & 1 \\ \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \end{vmatrix} = k(x-\alpha)(x-\beta)(x-\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$$

The leading term is $x^3\alpha^2\beta$. And in the expansion it is $kx^3(-\alpha^2)\beta$.

$\therefore \quad k = -1$ Hence the expansion $= -(x-\alpha)(x-\beta)(x-\gamma)(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$ **Proved.**

EXERCISE 4.5

1. Evaluate, without expanding $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}$ **Ans.** $(a-b)(b-c)(c-a)(1+abc)$

2. Without expanding, show that

$$\Delta = \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} = 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x).$$

3. Show (without expanding) that

$$\begin{vmatrix} bc & a^2 & a^2 \\ b^2 & ca & b^2 \\ c^2 & c^2 & ab \end{vmatrix} = \begin{vmatrix} bc & ab & ca \\ ab & ca & bc \\ ca & bc & ab \end{vmatrix} = -\frac{1}{2}(ab+bc+ca)[(ab-bc)^2+(bc-ca)^2+(ca-ab)^2]$$

4.9 PIVOTAL CONDENSATION METHOD

The condensation process of reducing n^{th} order determinant to $(n-1)^{\text{th}}$ order determinant is shown below :

Consider n^{th} order determinant
$$D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 & \dots & \dots \\ a_2 & b_2 & c_2 & d_2 & \dots & \dots \\ a_3 & b_3 & c_3 & d_3 & \dots & \dots \\ a_4 & b_4 & c_4 & d_4 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & b_n & c_n & d_n & \dots & \dots \end{vmatrix}$$

Add such a multiple of first column in the other columns so that at the places of b_1, c_1, d_1 ...., we get zero. Hence substracting $\frac{b_1}{a_1}, \frac{c_1}{a_1}, \frac{d_1}{a_1}, \dots,$ times the first column from the 2nd, 3rd, 4th... columns respectively, we get

$$D = \begin{vmatrix} a_1 & 0 & 0 & 0 & \dots & \dots \\ a_2 & b_2 - \frac{b_1}{a_1}.a_2 & c_2 - \frac{c_1}{a_1}a_2 & d_2 - \frac{d_1}{a_1}.a_2 & \dots & \dots \\ a_3 & b_3 - \frac{b_1}{a_1}.a_3 & c_3 - \frac{c_1}{a_1}a_3 & d_3 - \frac{d_1}{a_1}.a_3 & \dots & \dots \\ a_4 & b_4 - \frac{b_1}{a_1}.a_4 & c_4 - \frac{c_1}{a_1}a_4 & d_4 - \frac{d_1}{a_1}.a_4 & \dots & \dots \\ & \dots & \dots & \dots & & \\ & \dots & \dots & \dots & & \\ a_n & b_n - \frac{b_1}{a_1}.a_n & c_n - \frac{c_1}{a_1}.a_n & d_n - \frac{d_1}{a_1}.a_n & \dots & \dots \end{vmatrix}$$

$$D = a_1 \begin{vmatrix} b_2 - \frac{b_1}{a_1}.a_2 & c_2 - \frac{c_1}{a_1}a_2 & d_2 - \frac{d_1}{a_1}.a_2 & \cdots & \cdots \\ b_3 - \frac{b_1}{a_1}.a_3 & c_3 - \frac{c_1}{a_1}a_3 & d_3 - \frac{d_1}{a_1}.a_3 & \cdots & \cdots \\ b_4 - \frac{b_1}{a_1}.a_4 & c_4 - \frac{c_1}{a_1}a_4 & d_4 - \frac{d_1}{a_1}.a_4 & \cdots & \cdots \\ \cdots & \cdots & \cdots & & \\ \cdots & \cdots & \cdots & & \\ b_n - \frac{b_1}{a_1}.a_n & c_n - \frac{c_1}{a_1}.a_n & d_n - \frac{d_1}{a_1}.a_n & \cdots & \cdots \end{vmatrix} \quad \begin{bmatrix}\text{On expanding along} \\ \text{the first row}\end{bmatrix}$$

Which is a determinant of $(n-1)$th order. Now,

$$D = a_1 \begin{vmatrix} \frac{a_1b_2 - b_1a_2}{a_1} & \frac{a_1c_2 - c_1a_2}{a_1} & \frac{a_1d_2 - d_1a_2}{a_1} & \cdots & \cdots \\ \frac{a_1b_3 - b_1a_3}{a_1} & \frac{a_1c_3 - c_1a_3}{a_1} & \frac{a_1d_3 - d_1a_3}{a_1} & \cdots & \cdots \\ \frac{a_1b_4 - b_1a_4}{a_1} & \frac{a_1c_4 - c_1a_4}{a_1} & \frac{a_1d_4 - d_1a_4}{a_1} & \cdots & \cdots \\ \cdots & \cdots & \cdots & & \\ \cdots & & \cdots & & \\ \frac{a_1b_n - b_1a_n}{a_1} & \frac{a_1c_n - c_1a_n}{a_1} & \frac{a_1d_n - d_1a_n}{a_1} & \cdots & \end{vmatrix}$$

$$D = a_1 . \frac{1}{(a_1)^{n-1}} \begin{vmatrix} a_1b_2 - b_1a_2 & a_1c_2 - c_1a_2 & a_1d_2 - d_1a_2 & \cdots & \cdots \\ a_1b_3 - b_1a_3 & a_1c_3 - c_1a_3 & a_1d_3 - d_1a_3 & \cdots & \cdots \\ a_1b_4 - b_1a_4 & a_1c_4 - c_1a_4 & a_1d_4 - d_1a_4 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_1b_n - b_1a_n & a_1c_n - c_1a_n & a_1d_n - d_1a_n & \cdots & \cdots \end{vmatrix}$$

as the determinant is of $(n-1)^{\text{th}}$ order and $\frac{1}{a_1}$ is common in every row (or column)

$$= \frac{1}{(a_1)^{n-2}} \begin{vmatrix} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix} & \cdots & \cdots \\ \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_3 & d_3 \end{vmatrix} & \cdots & \cdots \\ \begin{vmatrix} a_1 & b_1 \\ a_4 & b_4 \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_4 & c_4 \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_4 & d_4 \end{vmatrix} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \begin{vmatrix} a_1 & b_1 \\ a_n & b_n \end{vmatrix} & \begin{vmatrix} a_1 & c_1 \\ a_n & c_n \end{vmatrix} & \begin{vmatrix} a_1 & d_1 \\ a_n & d_n \end{vmatrix} & \cdots & \cdots \end{vmatrix}$$

Thus, the n^{th} order determinant is condensed to $(n-1)$th order determinant. Repeated application of this method ultimately results in a determinant of 2nd order which can be evaluated.

It is obvious that the leading element a_1 behaves like a pivot in the condensation process (*i.e.*, reduction from n to $(n-1)$) and hence the method is pivotal condensation.

If the leading element is zero, it can be made non-zero by interchanging the columns.

Example 46. Condense the following determinants to second order and hence evaluate them:

$$(i)\quad \begin{vmatrix} 10 & 2 & -3 \\ 5 & 12 & 15 \\ 7 & -6 & 4 \end{vmatrix} \qquad (ii)\quad D = \begin{vmatrix} 2 & 1 & 3 & 5 \\ 4 & -2 & 7 & 6 \\ -8 & 3 & 1 & 0 \\ 5 & 7 & 2 & -6 \end{vmatrix}$$

Solution. (*i*) Using the leading element as pivot, we get

$$D = \frac{1}{(10)^{3-2}} \begin{vmatrix} 120-10 & 150+15 \\ -60-14 & 40+21 \end{vmatrix} \quad [\because \text{ order} = 3]$$

$$\Rightarrow\ D = \frac{1}{10} \begin{vmatrix} 110 & 165 \\ -74 & 61 \end{vmatrix} = \frac{55}{10} \begin{vmatrix} 2 & 3 \\ -74 & 61 \end{vmatrix} = \frac{11}{2}(122+222) = \frac{11}{2} \times 344 = 11 \times 172 = \mathbf{1892}. \quad \textbf{Ans.}$$

$$(ii)\quad \frac{1}{(2)^{4-2}} \begin{vmatrix} -4-4 & 14-12 & 12-20 \\ 6+8 & 2+24 & 0+40 \\ 14-5 & 4-15 & -12-25 \end{vmatrix} \text{ as the order is 4.}$$

$$= \frac{1}{4} \begin{vmatrix} -8 & 2 & -8 \\ 14 & 26 & 40 \\ 9 & -11 & -37 \end{vmatrix} = \frac{2 \times 2}{4} \begin{vmatrix} -4 & 1 & -4 \\ 7 & 13 & 20 \\ 9 & -11 & -37 \end{vmatrix} = \frac{1}{(-4)^{3-2}} \begin{vmatrix} -52-7 & -80+28 \\ 44-9 & 148+36 \end{vmatrix}$$

$$= -\frac{1}{4} \begin{vmatrix} -59 & -52 \\ 35 & 184 \end{vmatrix} = \frac{4}{4} \begin{vmatrix} 59 & 52 \\ 35 & 46 \end{vmatrix} = 59 \times 46 - 13 \times 35 = \mathbf{2259}. \quad \textbf{Ans.}$$

Example 47. Condense and hence evaluate the determinant, $\begin{vmatrix} 0 & 4 & 1 & 2 \\ 5 & 3 & 7 & 8 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 5 & 5 \end{vmatrix}$

Solution. As the leading element is zero, hence interchanging the 1st and second columns, we get

$$\begin{vmatrix} 0 & 4 & 1 & 2 \\ 5 & 3 & 7 & 8 \\ 4 & 1 & 2 & 3 \\ 1 & 2 & 5 & 5 \end{vmatrix} = -\begin{vmatrix} 4 & 0 & 1 & 2 \\ 3 & 5 & 7 & 8 \\ 1 & 4 & 2 & 3 \\ 2 & 1 & 5 & 5 \end{vmatrix} = -\frac{1}{4^2} \begin{vmatrix} 20-0 & 28-3 & 32-6 \\ 16-0 & 8-1 & 12-2 \\ 4-0 & 20-2 & 20-4 \end{vmatrix} = -\frac{1}{16} \begin{vmatrix} 20 & 25 & 26 \\ 16 & 7 & 10 \\ 4 & 18 & 16 \end{vmatrix}$$

$$= -\frac{4 \times 2}{16} \begin{vmatrix} 5 & 25 & 13 \\ 4 & 7 & 5 \\ 1 & 18 & 8 \end{vmatrix} = -\frac{1}{2} \cdot \frac{1}{5} \begin{vmatrix} 35-100 & 25-52 \\ 90-25 & 40-13 \end{vmatrix} = -\frac{1}{10} \begin{vmatrix} -65 & -27 \\ 65 & 27 \end{vmatrix} = \frac{1}{10} \begin{vmatrix} 65 & 27 \\ 65 & 27 \end{vmatrix} = 0.$$

Example 48. By condensing the given determinant evaluate x, $\begin{vmatrix} x-1 & 7 & 9 & 3 \\ 1 & 0 & 2 & 5 \\ 2x+2 & 6 & 8 & 3 \\ -2 & 1 & 1 & 0 \end{vmatrix} = 0.$

Solution. $$D = \begin{vmatrix} x-1 & 7 & 9 & 3 \\ 1 & 0 & 2 & 5 \\ 2x+2 & 6 & 8 & 3 \\ -2 & 1 & 1 & 0 \end{vmatrix} = -\begin{vmatrix} 7 & x-1 & 9 & 3 \\ 0 & 1 & 2 & 5 \\ 6 & 2x+2 & 8 & 3 \\ 1 & 2 & 1 & 0 \end{vmatrix}$$

$$= -\frac{1}{7^2}\begin{vmatrix} 7 & 14 & 35 \\ 14x+14-6x+6 & 56-54 & 21-18 \\ -14-x+1 & 7-9 & 0-3 \end{vmatrix} = -\frac{1}{7^2}\times 7\begin{vmatrix} 1 & 2 & 5 \\ 8x+20 & 2 & 3 \\ -x-13 & -2 & -3 \end{vmatrix}$$

$$= +\frac{2}{7}\begin{vmatrix} 1 & 1 & 5 \\ 8x+20 & 1 & 3 \\ x+13 & 1 & 3 \end{vmatrix} = \frac{2}{7}\begin{vmatrix} 1-8x-20 & 3-40x-100 \\ 1-x-13 & 3-5x-65 \end{vmatrix}$$

$$= \frac{2}{7}\begin{vmatrix} -8x-19 & -40x-97 \\ -x-12 & -5x-62 \end{vmatrix} = \frac{2}{7}\left[(8x+19)(5x+62)-(40x+97)(x+12)\right]$$

$$= \frac{2}{7}\left[40x^2+95x+496x+1178-40x^2-97x-480x-1164\right] = \frac{2}{7}[14x+14] = 4x+4$$

Thus $4x+4=0 \quad \Rightarrow \quad x+1=0 \quad \Rightarrow x=-1$ **Ans.**

Exercise 4.6

Using the leading element as pivots, codense the following determinants to second order and hence evaluate them.

1. $\begin{vmatrix} 1 & 3 & 7 \\ 4 & 9 & 1 \\ 2 & 7 & 6 \end{vmatrix}$ **Ans.** 51

2. $\begin{vmatrix} 2 & 0 & 2 \\ 3 & 7 & 4 \\ -2 & -5 & 1 \end{vmatrix}$ **Ans.** 52

3. $\begin{vmatrix} 5 & 2 & 7 \\ 9 & 1 & 10 \\ -2 & 3 & 4 \end{vmatrix}$ **Ans.** – 39

4. $\begin{vmatrix} 1 & 2 & 1 & 3 \\ 3 & 4 & 2 & 5 \\ 6 & 1 & 7 & 1 \\ 4 & 3 & 9 & 2 \end{vmatrix}$ **Ans.** 75

5. $\begin{vmatrix} 4 & -2 & 3 & 0 \\ 1 & 0 & 2 & 7 \\ -5 & 1 & 6 & 1 \\ 2 & 3 & 5 & -4 \end{vmatrix}$ **Ans.** –1334

6. $\begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 7 & 2 & 1 & 1 \\ 9 & 4 & 1 & 2 & 3 \\ 8 & 1 & 3 & 7 & 2 \\ 4 & 2 & 0 & 3 & 1 \end{vmatrix}$ **Ans.** –2276

7. Condense the following determinant and hence evaluate x,

$\begin{vmatrix} 3 & 2 & 1 & 5 \\ 4 & 7 & 6 & 2 \\ 2 & 1 & x-1 & -4 \\ 5 & 3 & 4 & 1 \end{vmatrix} = 0.$ **Ans.** $x = 4$

4.10 CONJUGATE ELEMENTS

Two equidistant elements lying on a line perpendicular to the leading diagonal are said to be conjugate.

In the determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, a_2, b_1; a_3, c_1; b_3, c_2; are pairs of conjugate elements.

4.11. SPECIAL TYPES OF DETERMINANTS

(*i*) Orthosymmetric Determinant. If every element of the leading diagonal is the same and the conjugate elements are equal, then the determinant is said to be orthosymmetric determinant.

$$\begin{vmatrix} a & h & g \\ h & a & f \\ g & f & a \end{vmatrix}$$

(*ii*) Skew-Symmetric Determinant. If the elements of the leading diagonal are all zero and every other element is equal to its conjugate with sign changed, the determinant is said to be Skew-symmetric.

$$\begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix}$$

Property 1. A Skew-symmetric determinant of odd order vanishes.

Example 49. *Prove that* $\Delta = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$

Solution. Taking out (– 1) common from each of the three columns

$$\Delta = (-1)^3 \begin{vmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{vmatrix}$$

Changing rows into columns $\Delta = (-1)^3 \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = (-1)^3\ \Delta = -\Delta$

or $\quad 2\Delta = 0$ or $\Delta = 0$ **Proved.**

Property 2. A skew-symmetric determinant of even order is a perfect square.

Example 50. Prove that $\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix} = (ax - by + cz)^2$

Solution. The given determinant is, by multiplying column 2 by a $= \dfrac{1}{a} \begin{vmatrix} 0 & ax & y & z \\ -x & 0 & c & b \\ -y & -ac & 0 & a \\ -z & -ab & -a & 0 \end{vmatrix}$

On expanding by column 2, we get

$$= \frac{-(ax - by + cz)}{a} \begin{vmatrix} -x & c & b \\ -y & 0 & a \\ -z & -a & 0 \end{vmatrix} = \frac{(ax - by + cz)}{a} \begin{vmatrix} x & c & b \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$

$$= \frac{(ax-by+cz)}{a\times a}\begin{vmatrix} ax & ac & b \\ y & 0 & a \\ z & -a & 0 \end{vmatrix} = \frac{(ax-by+cz)}{a^2}\begin{vmatrix} ax-by+cz & ac-ac & ab-ab \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$

$[\because R_1 \to R_1 - bR_2 + cR_3]$

$$= \frac{(ax-by+cz)}{a^2}\begin{vmatrix} ax-by+cz & 0 & 0 \\ y & 0 & a \\ z & -a & 0 \end{vmatrix} = \frac{(ax-by+cz)}{a^2}(ax-by+cz)(a^2)$$

$= (ax - by + cz)^2$

Proved.

4.12 LAPLACE METHOD FOR THE EXPANSION OF A DETERMINANT IN TERMS OF FIRST TWO ROWS

(*i*) Make all possible determinants from first two rows by taking any two columns.

(*ii*) Multiply each of them by corresponding determinant which is left by suppressing the rows and columns intersecting at them.

(*iii*) Add them with proper signs.

Here we count the number of movements of columns of the determinant by shifting to the place of the first determinant. If the number of movement is odd then negative sign, if even then positive sign.

Example 51. Expand the determinant $\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ by Laplace method.

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}\begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix} + \begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}\begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix} + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}\begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix}$$

$$- \begin{vmatrix} b_1 & d_1 \\ b_2 & d_2 \end{vmatrix}\begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix} + \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix}\begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix}$$

Explanation : $\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

Now the c column being 3rd can be made 2nd by one movement of column; "*a*" column is in the position of first column so that the total number of movements is one i.e. odd; hence the sign will be –ve.

Example 52. Expand the following determinant by Laplace method : $\begin{vmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$

Solution . $\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - \begin{vmatrix} a_1 & 0 \\ a_2 & 0 \end{vmatrix}\begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix}$

$$+ \begin{vmatrix} a_1 & 0 \\ a_2 & 0 \end{vmatrix}\begin{vmatrix} b_3 & c_3 \\ b_4 & c_4 \end{vmatrix} + \begin{vmatrix} b_1 & 0 \\ b_2 & 0 \end{vmatrix}\begin{vmatrix} a_3 & d_3 \\ a_4 & d_4 \end{vmatrix} - \begin{vmatrix} b_1 & 0 \\ b_2 & 0 \end{vmatrix}\begin{vmatrix} a_3 & c_3 \\ a_4 & c_4 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix}\begin{vmatrix} a_3 & b_3 \\ a_4 & b_4 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} = (a_1b_2 - \alpha_2b_1)(c_3d_4 - c_4d_3)$$

Ans.

4.13. APPLICATION OF DETERMINANTS

Area of triangle. We know that the area of a triangle, whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\left[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)\right]$$

$$= \frac{1}{2}\left[x_1\begin{vmatrix} y_2 & 1 \\ y_3 & 1 \end{vmatrix} - x_2\begin{vmatrix} y_1 & 1 \\ y_3 & 1 \end{vmatrix} + x_3\begin{vmatrix} y_1 & 1 \\ y_2 & 1 \end{vmatrix}\right] = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note. Since area is always a positive quantity, therefore we always take the absolute value of the determinant for the area.

Condition of collinearity of three points. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be three points. Then,

A, B, C are collinear $\Leftrightarrow$ area of triangle $ABC = 0$

$$\Leftrightarrow \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad \Leftrightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Proved.

Example 53. *Using determinants, find the area of the triangle with vertices (– 3, 5), (3, – 6) and (7, 2).*

Solution. The area of the given triangle $= \frac{1}{2}\begin{vmatrix} -3 & 5 & 1 \\ 3 & -6 & 1 \\ 7 & 2 & 1 \end{vmatrix}$

Operate : $R_1 \to R_1 - R_2$; $R_2 \to R_2 - R_3$ $= \frac{1}{2}\begin{vmatrix} -6 & 11 & 0 \\ -4 & -8 & 0 \\ 7 & 2 & 1 \end{vmatrix}$

Expand by C_3 $= \frac{1}{2}.1.\begin{vmatrix} -6 & 11 \\ -4 & -8 \end{vmatrix} = \frac{1}{2}(48 + 44) = 46$ sq. units **Ans.**

Example 54. *Using determinants, show that the points (11, 7), (5, 5) and (– 1, 3) are collinear.*

Solution. The area of the triangle formed by the given points $= \frac{1}{2}\begin{vmatrix} 11 & 7 & 1 \\ 5 & 5 & 1 \\ -1 & 3 & 1 \end{vmatrix}$

Operate : $R_1 \to R_1 - R_2$; $R_2 \to R_2 - R_3$

$$= \frac{1}{2}\begin{vmatrix} 6 & 2 & 0 \\ 6 & 2 & 0 \\ -1 & 3 & 1 \end{vmatrix} = \frac{1}{2}.0 = 0. \qquad (\because R_1 \text{ and } R_2 \text{ are identical})$$

$\Rightarrow$ The three given points are collinear. **Proved.**

Example 55. *Using determinants, find the area of the triangle whose vertices are (1, 4) (2, 3) and (– 5, – 3). Are the given points collinear?*

Solution. Area of the required triangle $= \frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 2 & 3 & 1 \\ -5 & -3 & 1 \end{vmatrix}$

$$= \frac{1}{2}[1(3+3) - 4(2+5) + 1(-6+15)] = \frac{1}{2}[6 - 28 + 9] = 13/2 \neq 0$$

Hence, the given points are not collinear. **Ans.**

EXERCISE 4.7

Using determinants, find the area of the triangle with vertices:

1. $(2,-7), (1, 3), (10, 8)$. **Ans.** $A = \frac{95}{2}$ 2. $(-2, 4), (2, -6)$ and $(5, 4)$. **Ans.** Area = 35
3. $(-1, -3), (2, 4)$ and $(3, -1)$. **Ans.** A = 11 4. $(1, -1), (2, 4)$ and $(-3, 5)$. **Area = 13**
5. Using determinants, show that the points $(3, 8), (-4, 2)$ and $(10, 14)$ are collinear.
6. Find the value of α, so that the points $(1, -5), (-4, 5)$ and $(\alpha, 7)$ are collinear. **Ans.** $\alpha = -5$
7. Find the value of x, if the area of Δ is 35 square cms with vertices $(x, 4), (2, -6), (5, 4)$. **Ans.** $x = -2, 12$
8. Using determinants find the value of k, so that the points $(k, 2-2k), (-k+1, 2k)$ and $(-4-k, 6-2k)$ may be collinear. **Ans.** $k = -1, \frac{1}{2}$
9. If the points $(x, -2), (5, 2)$ and $(8, 8)$ are collinear, find x using determinants. **Ans.** $x = 3$
10. If the points $(3, -2), (x, 2)$ and $(8, 8)$ are collinear, find x using determinants. **Ans.** $x = 1$

4.14. SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS BY DETERMINANTS (CRAMER'S RULE)

Let us solve the following equations.

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

Let
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{or} \quad x\,D = \begin{vmatrix} a_1 x & b_1 & c_1 \\ a_2 x & b_2 & c_2 \\ a_3 x & b_3 & c_3 \end{vmatrix}$$

Multiplying the 2nd column by y and 3rd column by z and adding to the 1st column, we get

$$x\,D = \begin{vmatrix} a_1 x + b_1 y + c_1 z & b_1 & c_1 \\ a_2 x + b_2 y + c_2 z & b_2 & c_2 \\ a_3 x + b_3 y + c_3 z & b_3 & c_3 \end{vmatrix} \Rightarrow x\,D = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Rightarrow \quad x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_1 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}} = \frac{D_1}{D} \quad \text{Similarly,} \quad y = \frac{D_2}{D} = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$z = \frac{D_3}{D} = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$x = \frac{D_1}{D}, \qquad y = \frac{D_2}{D}, \qquad z = \frac{D_3}{D}$$ **Ans**

Example 56. *Solve the following system of equations using Cramer's rule :*

$$5x - 7y + z = 11$$
$$6x - 8y - z = 15$$
$$3x + 2y - 6z = 7$$

Solution. The given equations are

$$5x - 7y + z = 11$$
$$6x - 8y - z = 15$$
$$3x + 2y - 6z = 7$$

Here $D = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix} = 5\,(48 + 2) + 7\,(-36 + 3) + 1\,(12 + 24) = 55\ (\neq 0)$

$$D_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix} = 11\,(48 + 2) + 7\,(-90 + 7) + 1\,(30 + 56) = 55$$

$$D_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix} = 5\,(-90 + 7) - 11\,(-36 + 3) + 1\,(42 - 45) = -55$$

$$D_3 = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix} = 5\,(-56 - 30) + 7\,(42 - 45) + 11\,(12 + 24) = -55$$

By Cramer's Rule $x = \frac{D_1}{D} = \frac{55}{55} = 1$, $\quad y = \frac{D_2}{D} = \frac{-55}{55} = -1$, $\quad z = \frac{D_3}{D} = \frac{-55}{55} = -1$

Hence, $x = 1$, $\quad y = -1$, $\quad z = -1$ **Ans.**

Example 57. *Solve, by determinants, the following set of simultaneous equations :*

$$5x - 6y + 4z = 15$$
$$7x + 4y - 3z = 19$$
$$2x + y + 6z = 46$$

Solution. $D = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix} = 419$

$$D_1 = \begin{vmatrix} 15 & -6 & 4 \\ 19 & 4 & -3 \\ 46 & 1 & 6 \end{vmatrix} = 1257, \quad D_2 = \begin{vmatrix} 5 & 15 & 4 \\ 7 & 19 & -3 \\ 2 & 46 & 6 \end{vmatrix} = 1676, \quad D_3 = \begin{vmatrix} 5 & -6 & 15 \\ 7 & 4 & 19 \\ 2 & 1 & 46 \end{vmatrix} = 2514$$

By Cramer's Rule: $x = \frac{D_1}{D} = \frac{1257}{419} = 3$, $\quad y = \frac{D_2}{D} = \frac{1676}{419} = 4$, $\quad z = \frac{D_3}{D} = \frac{2514}{419} = 6.$

Hence, $x = 3$, $\quad y = 4$, $\quad z = 6$ **Ans.**

Example 58. *Solve the following system of equations using Cramer's Rule :*

$$2x - 3y + 4z = -9$$
$$-3x + 4y + 2z = -12$$
$$4x - 2y - 3z = -3$$

Solution. The given equations are

$$2x - 3y + 4z = -9$$
$$-3x + 4y + 2z = -12$$
$$4x - 2y - 3z = -3$$

Here $D = \begin{vmatrix} 2 & -3 & 4 \\ -3 & 4 & 2 \\ 4 & -2 & -3 \end{vmatrix} = 2(-12+4) + 3(9-8) + 4(6-16) = -53$

$$D_1 = \begin{vmatrix} -9 & -3 & 4 \\ -12 & 4 & 2 \\ -3 & -2 & -3 \end{vmatrix} = -9(-12+4) + 3(36+6) + 4(24+12) = -342$$

$$D_2 = \begin{vmatrix} 2 & -9 & 4 \\ -3 & -12 & 2 \\ 4 & -3 & -3 \end{vmatrix} = 2(36+6) + 9(9-8) + 4(9+48) = -321$$

$$D_3 = \begin{vmatrix} 2 & -3 & -9 \\ -3 & 4 & -12 \\ 4 & -2 & -3 \end{vmatrix} = 2(-12-24) + 3(9+48) - 9(6-16) = -189$$

By Cramer's Rule,

$$x = \frac{D_1}{D} = \frac{-342}{-53} = \frac{342}{53}, \quad y = \frac{D_2}{D} = \frac{-321}{-53} = \frac{321}{53}, \quad z = \frac{D_3}{D} = \frac{-189}{-53} = \frac{189}{53}$$

Hence, $x = \frac{342}{53}$, $\quad y = \frac{321}{53}$, $\quad z = \frac{189}{53}$ **Ans.**

Example 59. *Solve the following system of equations by using determinants :*

$$x + y + z = 1$$
$$ax + by + cz = k$$
$$a^2x + b^2y + c^2z = k^2$$

Solution. We have $D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \text{[Applying } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1]$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & b+a & c+a \end{vmatrix}$$

$$= (b-a)(c-a).1. \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix} \qquad \text{[Expanding along } R_1]$$

$$= (b-a)(c-a)(c+a-b-a)$$

$$= (b-c)(c-a)(a-b) \qquad ...(1)$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix} = (b-c)(c-k)(k-b) \qquad \text{[Replacing } a \text{ by } k \text{ in (1)]}$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & k & c \\ a^2 & k^2 & c^2 \end{vmatrix} = (k-c)(c-a)(a-k) \qquad \text{[Replacing } b \text{ by } k \text{ in (1)]}$$

and $$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & k \\ a^2 & b^2 & k^2 \end{vmatrix} = (a-b)(b-k)(k-a) \qquad \text{[Replacing } c \text{ by } k \text{ in (1)]}$$

$\therefore$ $$x = \frac{D_1}{D} = \frac{(b-c)(c-k)(k-b)}{(b-c)(c-a)(a-b)}, \quad y = \frac{D_2}{D} = \frac{(k-c)(c-a)(a-k)}{(b-c)(c-a)(a-b)}$$

and $$z = \frac{D_3}{D} = \frac{(a-b)(b-k)(k-a)}{(a-b)(b-c)(c-a)}$$

Hence, $$x = \frac{(c-k)(k-b)}{(c-a)(a-b)}, \quad y = \frac{(k-c)(a-k)}{(b-c)(a-b)} \text{ and } z = \frac{(b-k)(k-a)}{(b-c)(c-a)} \qquad \textbf{Ans.}$$

Example 60. *The sum of three numbers is 6. If we multiply the third number by 2 and add the first number to the result, we get 7. By adding second and third numbers to three times the first number we get 12. Use determinants to find the numbers.*

Solution. Let the three numbers be x, y and z. Then, from the given conditions, we have

$$\left.\begin{aligned} x+y+z &= 6 \\ x+2z &= 7 \\ 3x+y+z &= 12 \end{aligned}\right\} \quad \text{or} \quad \left\{\begin{aligned} x+y+z &= 6 \\ x+0.y+2z &= 7 \\ 3x+y+z &= 12 \end{aligned}\right.$$

Here, $$D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 1(0-2) - 1(1-6) + 1(1-0) = -2+5+1 = 4$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 7 & 0 & 2 \\ 12 & 1 & 1 \end{vmatrix} = 6(0-2) - 1(7-24) + (7-0) = -12+17+7 = 12$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 7 & 2 \\ 3 & 12 & 1 \end{vmatrix} = 1(7-24) - 6(1-6) + 1(12-21) = -17 + 30 - 9 = 4$$

and $$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 0 & 7 \\ 3 & 1 & 12 \end{vmatrix} = 1(0-7) - 1(12-21) + 6(1-0) = -7 + 9 + 6 = 8$$

$$\therefore \quad x = \frac{D_1}{D} = \frac{12}{4} = 3, \quad y = \frac{D_2}{D} = \frac{4}{4} = 1, \text{ and } z = \frac{D_3}{D} = \frac{8}{4} = 2$$

Thus, the three numbers are 3, 1 and 2. **Ans.**

EXERCISE 4.8

Using Cramer's Rule, solve the following system of equations :

1. $2x - 3y = 7$, $7x - 3y = 10$ **Ans.** $x = \frac{3}{5}$, $y = -\frac{29}{15}$

2. $2x + y = 1$, $x - 2y = 8$ **Ans.** $x = 2, y = -3$

3. $2x + 3y = 10$, $x + 6y = 4.$ **Ans.** $x = \frac{16}{3}$, $y = -\frac{2}{9}$

4. $5x + 2y = 3$, $3x + 2y = 5.$ **Ans.** $x = -1,\ y = 4$

5. $7x - 2y = -7$, $2x - y = 1.$ **Ans.** $x = -3, y = -7$

6. $x - 2y = 4$, $-3x + 5y = -7$ **Ans.** $x = -6,\ y = -5$

7. $x - 4y - z = 11$, $2x - 5y + 2z = 39$, $-3x + 2y + z = 1.$ **Ans.** $x = -1,\ y = -5,\ z = 8$

8. $x + 3y - 2z = 5$, $2x + y + 4z = 8$, $6x + y - 3z = 5.$ **Ans.** $x = 1,\ y = 2,\ z = 1$

9. $x + 2y + 5z = 23$, $3x + y + 4z = 26$, $6x + y + 7z = 47$ **Ans.** $x = 4,\ y = 2,\ z = 3$

10. $x + y + z = 1$, $3x + 5y + 6z = 4$, $9x + 2y - 36x = 17$ **Ans.** $x = \frac{1}{3}$, $y = 1$, $z = -\frac{1}{3}$

11. $2y - z = 0$, $x + 3y = -4$, $3x + 4y = 3$ **Ans.** $x = 5, y = -3, z = -6$

12. $x + y + z = -1$, $x + 2y + 3z = -4$, $x + 3y + 4z = -6$ **Ans.** $x = 1,\ y = -1,\ z = -1$

13. $x + y + z = 1$
$x + 2y + 3z = k$
$1^2x + 2^2y + 3^2z = k^2$ **Ans.** $x = \frac{(2-k)(3-k)}{2}$, $y = \frac{(1-k)(3-k)}{-1}$, $z = \frac{(1-k)(2-k)}{2}$

14. Show that there are three real values of λ for which the equations:
$(a - \lambda)x + by + cz = 0$
$bx + (c - \lambda)y + az = 0$
$cx + ay + (b - \lambda)z = 0$
are simultaneously true, and that the product of these values of λ is $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

15. Solve the following system of equations by using the Cramer's Rule

$x_1 + x_2 = 1$; $x_2 + x_3 = 0$; $x_3 + x_4 = 0$; $x_4 + x_5 = 0$; $x_5 + x_1 = 0$ (*A.M.I.E.T.E., Summer 2005*)

Ans. $x_1 = \frac{1}{2}$, $x_2 = \frac{1}{2}$, $x_3 = -\frac{1}{2}$, $x_4 = \frac{1}{2}$, $x_5 = -\frac{1}{2}$