College of Remote Sensing and Geopgysics

Dept. of Geophysics

CALCULUS I

Review

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Review : Functions

In this section we're going to make sure that you're familiar with functions and function notation. Both will appear in almost every section in a Calculus class and so you will need to be able to deal with them.

First, what exactly is a function? An equation will be a function if for any *x* in the domain of the equation (the domain is all the *x*'s that can be plugged into the equation) the equation will yield exactly one value of *y*.

This is usually easier to understand with an example.

Example 1 Determine if each of the following are functions.

(a) $y = x^2 + 1$

(b) $y^2 = x + 1$

Solution

(a) This first one is a function. Given an *x*, there is only one way to square it and then add 1 to the result. So, no matter what value of *x* you put into the equation, there is only one possible value of *y*.

(b) The only difference between this equation and the first is that we moved the exponent off the *x* and onto the *y*. This small change is all that is required, in this case, to change the equation from a function to something that isn't a function.

To see that this isn't a function is fairly simple. Choose a value of *x*, say $x=3$ and plug this into the equation.

$$y^2 = 3 + 1 = 4$$

Now, there are two possible values of *y* that we could use here. We could use $y = 2$ or $y = -2$. Since there are two possible values of *y* that we get from a single *x* this equation isn't a function.

Note that this only needs to be the case for a single value of *x* to make an equation not be a function. For instance we could have used $x=-1$ and in this case we would get a single *y* ($y=0$). However, because of what happens at $x=3$ this equation will not be a function.

Next we need to take a quick look at function notation. Function notation is nothing more than a fancy way of writing the *y* in a function that will allow us to simplify notation and some of our work a little.

Let's take a look at the following function.

$$y = 2x^2 - 5x + 3$$

Using function notation we can write this as any of the following.

$$\begin{array}{ll} f(x) = 2x^2 - 5x + 3 & g(x) = 2x^2 - 5x + 3 \\ h(x) = 2x^2 - 5x + 3 & R(x) = 2x^2 - 5x + 3 \\ w(x) = 2x^2 - 5x + 3 & y(x) = 2x^2 - 5x + 3 \end{array}$$

$$\vdots$$

Recall that this is NOT a letter times *x*, this is just a fancy way of writing *y*.

So, why is this useful? Well let's take the function above and let's get the value of the function at $x=-3$. Using function notation we represent the value of the function at $x=-3$ as $f(-3)$. Function notation gives us a nice compact way of representing function values.

Now, how do we actually evaluate the function? That's really simple. Everywhere we see an *x* on the right side we will substitute whatever is in the parenthesis on the left side. For our function this gives,

$$\begin{aligned} f(-3) &= 2(-3)^2 - 5(-3) + 3 \\ &= 2(9) + 15 + 3 \\ &= 36 \end{aligned}$$

All throughout a calculus course we will be finding roots of functions. A root of a function is nothing more than a number for which the function is zero. In other words, finding the roots of a function, *g(x)*, is equivalent to solving

$$g\left(x\right)=0$$

Example 3 Determine all the roots of $f\left(t\right)=9t^3-18t^2+6t$

Solution
So we will need to solve,

$$9t^3-18t^2+6t=0$$

First, we should factor the equation as much as possible. Doing this gives,

$$3t\left(3t^2-6t+2\right)=0$$

Next recall that if a product of two things are zero then one (or both) of them had to be zero. This means that,

$$3t=0 \qquad\qquad \text{OR,}$$

$$3t^2-6t+2=0$$

From the first it's clear that one of the roots must then be $t=0$. To get the remaining roots we will need to use the quadratic formula on the second equation. Doing this gives,

$$\begin{aligned} t &= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} \\ &= \frac{6 \pm \sqrt{12}}{6} \\ &= \frac{6 \pm \sqrt{(4)(3)}}{6} \\ &= \frac{6 \pm 2\sqrt{3}}{6} \\ &= \frac{3 \pm \sqrt{3}}{3} \\ &= 1 \pm \frac{1}{3}\sqrt{3} \\ &= 1 \pm \frac{1}{\sqrt{3}} \end{aligned}$$

In order to remind you how to simplify radicals we gave several forms of the answer.

To complete the problem, here is a complete list of all the roots of this function.

$$t = 0,\ t = \frac{3+\sqrt{3}}{3},\ \ t = \frac{3-\sqrt{3}}{3}$$

Note we didn't use the final form for the roots from the quadratic. This is usually where we'll stop with the simplification for these kinds of roots. Also note that, for the sake of the practice, we broke up the compact form for the two roots of the quadratic. You will need to be able to do this so make sure that you can.

This example had a couple of points other than finding roots of functions.

The first was to remind you of the quadratic formula. This won't be the last time that you'll need it in this class.

The second was to get you used to seeing "messy" answers. In fact, the answers in the above list are not that messy. However, most students come out of an Algebra class very used to seeing only integers and the occasional "nice" fraction as answers.

So, here is fair warning. In this class I often will intentionally make the answers look "messy" just to get you out of the habit of always expecting "nice" answers. In "real life" (whatever that is) the answer is rarely a simple integer such as two. In most problems the answer will be a decimal that came about from a messy fraction and/or an answer that involved radicals.

One of the more important ideas about functions is that of the **domain** and **range** of a function. In simplest terms the domain of a function is the set of all values that can be plugged into a function and have the function exist and have a real number for a value. So, for the domain we need to avoid division by zero, square roots of negative numbers, logarithms of zero and logarithms of negative numbers (if not familiar with logarithms we'll take a look at them a little later), *etc*. The range of a function is simply the set of all possible values that a function can take.

Let's find the domain and range of a few functions.

Example 4 Find the domain and range of each of the following functions.

(a) $f(x) = 5x - 3$ [Solution]

(b) $g(t) = \sqrt{4 - 7t}$ [Solution]

(c) $h(x) = -2x^2 + 12x + 5$ [Solution]

(d) $f(z) = |z - 6| - 3$ [Solution]

(e) $g(x) = 8$ [Solution]

In general determining the range of a function can be somewhat difficult. As long as we restrict ourselves down to "simple" functions, some of which we looked at in the previous example, finding the range is not too bad, but for most functions it can be a difficult process.

Because of the difficulty in finding the range for a lot of functions we had to keep those in the previous set somewhat simple, which also meant that we couldn't really look at some of the more complicated domain examples that are liable to be important in a Calculus course. So, let's take a look at another set of functions only this time we'll just look for the domain.

Example 5 Find the domain of each of the following functions.

(a) $f\left(x\right)=\dfrac{x-4}{x^2-2x-15}$ [Solution]

(b) $g\left(t\right)=\sqrt{6+t-t^2}$ [Solution]

(c) $h\left(x\right)=\dfrac{x}{\sqrt{x^2-9}}$ [Solution]

The next topic that we need to discuss here is that of **function composition**. The composition of *f(x)* and *g(x)* is

$$\left(f \circ g\right)\left(x\right) = f\left(g\left(x\right)\right)$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note as well that order is important here. Interchanging the order will usually result in a different answer.

Example 6 Given $f\left(x\right) = 3x^2 - x + 10$ and $g\left(x\right) = 1 - 20x$ find each of the following.

(a) $\left(f \circ g\right)\left(5\right)$ [Solution]

(b) $\left(f \circ g\right)\left(x\right)$ [Solution]

(c) $\left(g \circ f\right)\left(x\right)$ [Solution]

(d) $\left(g \circ g\right)\left(x\right)$ [Solution]

Solution

(a) $\left(f \circ g\right)\left(5\right)$

In this case we've got a number instead of an *x* but it works in exactly the same way.

$$\begin{aligned}\left(f \circ g\right)\left(5\right) &= f\left(g\left(5\right)\right)\\ &= f\left(-99\right) = 29512\end{aligned}$$

(b) $\left(f \circ g\right)\left(x\right)$

Let's work one more example that will lead us into the next section.

Example 7 Given $f\left(x\right)=3x-2$ and $g\left(x\right)=\frac{1}{3}x+\frac{2}{3}$ find each of the following.

(a) $\left(f\circ g\right)\left(x\right)$

(b) $\left(g\circ f\right)\left(x\right)$

Solution

(a)

$$\begin{aligned}\left(f\circ g\right)\left(x\right)&=f\left(g\left(x\right)\right)\\&=f\left(\frac{1}{3}x+\frac{2}{3}\right)\\&=3\left(\frac{1}{3}x+\frac{2}{3}\right)-2\\&=x+2-2=x\end{aligned}$$

(b)

$$\begin{aligned}\left(g\circ f\right)\left(x\right)&=g\left(f\left(x\right)\right)\\&=g\left(3x-2\right)\\&=\frac{1}{3}\left(3x-2\right)+\frac{2}{3}\\&=x-\frac{2}{3}+\frac{2}{3}=x\end{aligned}$$

In this case the two compositions were the same and in fact the answer was very simple.

$$\left(f\circ g\right)\left(x\right)=\left(g\circ f\right)\left(x\right)=x$$

This will usually not happen. However, when the two compositions are the same, or more specifically when the two compositions are both x there is a very nice relationship between the two functions. We will take a look at that relationship in the next section.

Review : Inverse Functions

In the last example from the previous section we looked at the two functions $f(x) = 3x - 2$ and $g(x) = \frac{x}{3} + \frac{2}{3}$ and saw that

$$(f \circ g)(x) = (g \circ f)(x) = x$$

and as noted in that section this means that there is a nice relationship between these two functions. Let's see just what that relationship is. Consider the following evaluations.

$$f(-1) = 3(-1) - 2 = -5 \qquad \Rightarrow \qquad g(-5) = \frac{-5}{3} + \frac{2}{3} = \frac{-3}{3} = -1$$

$$g(2) = \frac{2}{3} + \frac{2}{3} = \frac{4}{3} \qquad \Rightarrow \qquad f\left(\frac{4}{3}\right) = 3\left(\frac{4}{3}\right) - 2 = 4 - 2 = 2$$

In the first case we plugged $x = -1$ into $f(x)$ and got a value of -5. We then turned around and plugged $x = -5$ into $g(x)$ and got a value of -1, the number that we started off with.

In the second case we did something similar. Here we plugged $x = 2$ into $g(x)$ and got a value of $\frac{4}{3}$, we turned around and plugged this into $f(x)$ and got a value of 2, which is again the number that we started with.

Note that we really are doing some function composition here. The first case is really,

$$(g \circ f)(-1) = g\left[f(-1)\right] = g[-5] = -1$$

and the second case is really,

$$(f \circ g)(2) = f\left[g(2)\right] = f\left[\frac{4}{3}\right] = 2$$

Note as well that these both agree with the formula for the compositions that we found in the previous section. We get back out of the function evaluation the number that we originally plugged into the composition.

So, just what is going on here? In some way we can think of these two functions as undoing what the other did to a number. In the first case we plugged $x = -1$ into $f(x)$ and then plugged the result from this function evaluation back into $g(x)$ and in some way $g(x)$ undid what $f(x)$ had done to $x = -1$ and gave us back the original x that we started with.

Function pairs that exhibit this behavior are called **inverse functions**. Before formally defining inverse functions and the notation that we're going to use for them we need to get a definition out of the way.

A function is called **one-to-one** if no two values of *x* produce the same *y*. Mathematically this is the same as saying,

$$f\left(x_1\right) \neq f\left(x_2\right) \qquad \text{whenever} \qquad x_1 \neq x_2$$

So, a function is one-to-one if whenever we plug different values into the function we get different function values.

Now, be careful with the notation for inverses. The "-1" is NOT an exponent despite the fact that is sure does look like one! When dealing with inverse functions we've got to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

This is one of the more common mistakes that students make when first studying inverse functions.

The process for finding the inverse of a function is a fairly simple one although there are a couple of steps that can on occasion be somewhat messy. Here is the process

Finding the Inverse of a Function

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

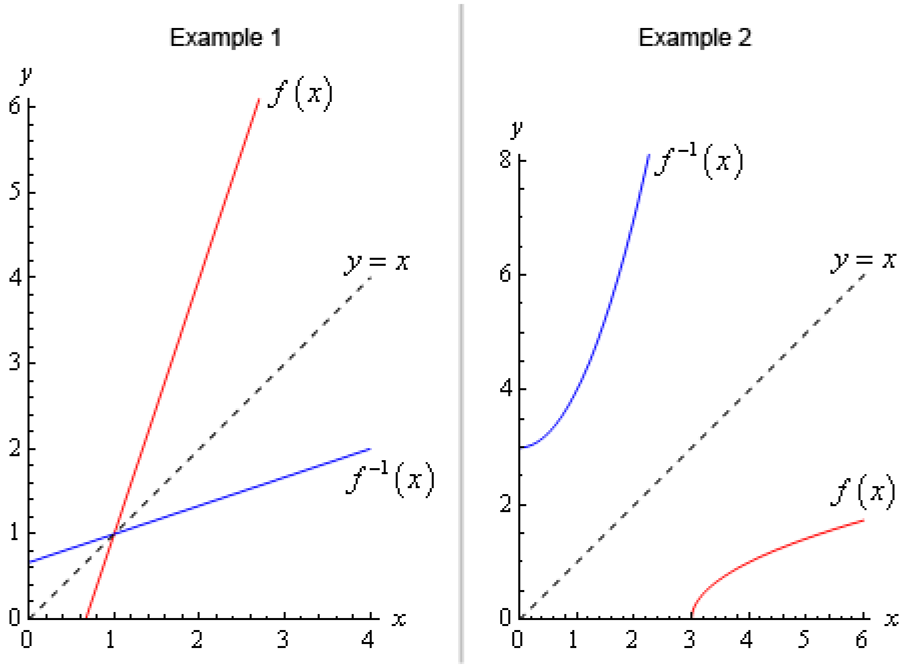
1. First, replace $f(x)$ with *y*. This is done to make the rest of the process easier.
2. Replace every *x* with a *y* and replace every *y* with an *x*.
3. Solve the equation from Step 2 for *y*. This is the step where mistakes are most often made so be careful with this step.
4. Replace *y* with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!
5. Verify your work by checking that $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

That's the process. Most of the steps are not all that bad but as mentioned in the process there are a couple of steps that we really need to be careful with since it is easy to make mistakes in those steps.

In the verification step we technically really do need to check that both $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ are true. For all the functions that we are going to be looking at in this course if one is true then the other will also be true. However, there are functions (they are beyond the scope of this course however) for which it is possible for only one of these to be true. This is brought up because in all the problems here we will be just checking one of them. We just need to always remember that technically we should check both.

There is one final topic that we need to address quickly before we leave this section. There is an interesting relationship between the graph of a function and the graph of its inverse.

Here is the graph of the function and inverse from the first two examples.



In both cases we can see that the graph of the inverse is a reflection of the actual function about the line $y = x$. This will always be the case with the graphs of a function and its inverse.

Review : Exponential Functions

In this section we're going to review one of the more common functions in both calculus and the sciences. However, before getting to this function let's take a much more general approach to things.

Let's start with $b > 0$, $b \neq 1$. An exponential function is then a function in the form,

$$f(x) = b^x$$

Note that we avoid $b = 1$ because that would give the constant function, $f(x) = 1$. We avoid $b = 0$ since this would also give a constant function and we avoid negative values of *b* for the following reason. Let's, for a second, suppose that we did allow *b* to be negative and look at the following function.

$$g(x) = (-4)^x$$

Let's do some evaluation.

$$g(2) = (-4)^2 = 16 \qquad\qquad g\left(\frac{1}{2}\right) = (-4)^{\frac{1}{2}} = \sqrt{-4} = 2i$$

So, for some values of *x* we will get real numbers and for other values of *x* we will get complex numbers. We want to avoid this and so if we require $b > 0$ this will not be a problem.

Let's take a look at a couple of exponential functions.

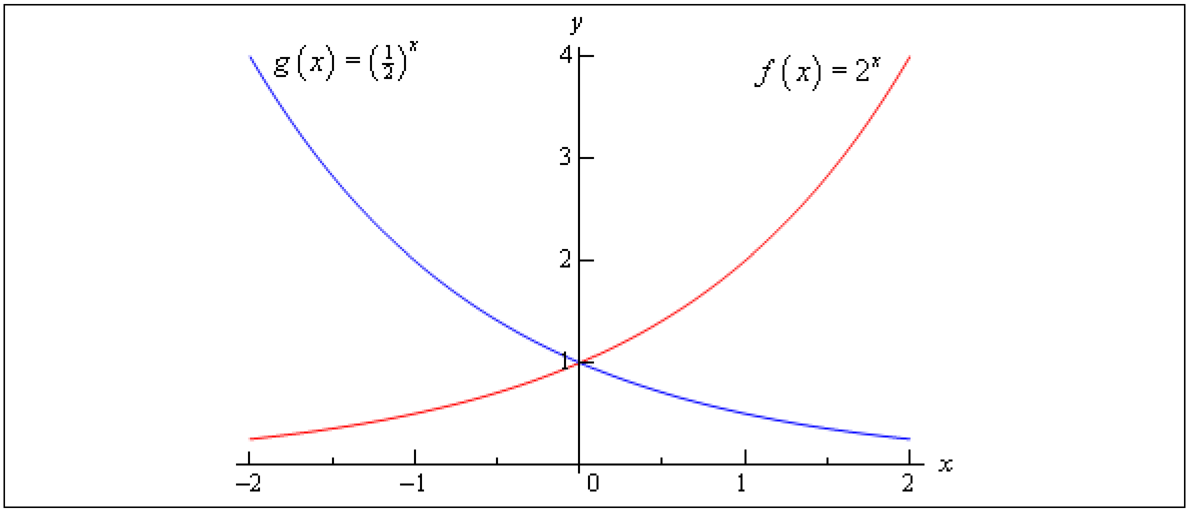
Example 1 Sketch the graph of $f(x) = 2^x$ and $g(x) = \left(\frac{1}{2}\right)^x$

Solution

Let's first get a table of values for these two functions.

x	*f(x)*	*g(x)*
-2	$f(-2) = 2^{-2} = \frac{1}{4}$	$g(-2) = \left(\frac{1}{2}\right)^{-2} = 4$
-1	$f(-1) = 2^{-1} = \frac{1}{2}$	$g(-1) = \left(\frac{1}{2}\right)^{-1} = 2$
0	$f(0) = 2^0 = 1$	$g(0) = \left(\frac{1}{2}\right)^0 = 1$
1	$f(1) = 2$	$g(1) = \frac{1}{2}$
2	$f(2) = 4$	$g(2) = \frac{1}{4}$

Here's the sketch of both of these functions.



This graph illustrates some very nice properties about exponential functions in general.

Properties of $f(x) = b^x$

1. $f(0) = 1$. The function will always take the value of 1 at $x = 0$.
2. $f(x) \neq 0$. An exponential function will never be zero.
3. $f(x) > 0$. An exponential function is always positive.
4. The previous two properties can be summarized by saying that the range of an exponential function is $(0, \infty)$.
5. The domain of an exponential function is $(-\infty, \infty)$. In other words, you can plug every x into an exponential function.
6. If $0 < b < 1$ then,
   a. $f(x) \to 0$ as $x \to \infty$
   b. $f(x) \to \infty$ as $x \to -\infty$
7. If $b > 1$ then,
   a. $f(x) \to \infty$ as $x \to \infty$
   b. $f(x) \to 0$ as $x \to -\infty$

These will all be very useful properties to recall at times as we move throughout this course (and later Calculus courses for that matter…).

There is a very important exponential function that arises naturally in many places. This function is called the **natural exponential function**. However, for must people this is simply the exponential function.

Definition : The **natural exponential function** is $f(x) = \mathbf{e}^x$ where,

$\mathbf{e} = 2.71828182845905\ldots$.

So, since $\mathbf{e} > 1$ we also know that $\mathbf{e}^x \to \infty$ as $x \to \infty$ and $\mathbf{e}^x \to 0$ as $x \to -\infty$.

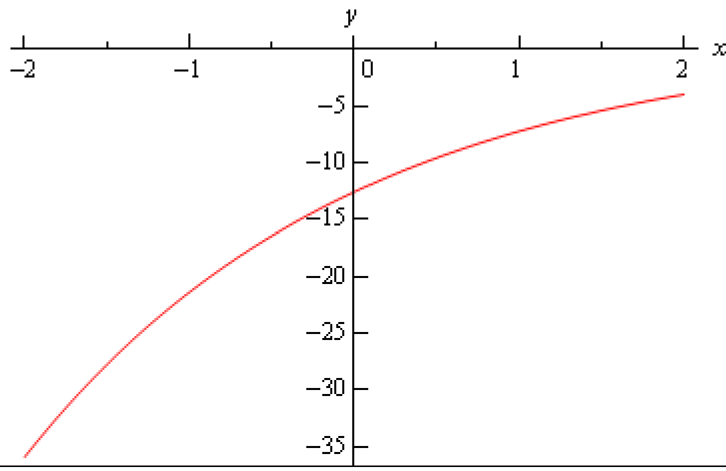
Let's take a quick look at an example.

Example 2 Sketch the graph of $h(t) = 1 - 5\mathbf{e}^{1-\frac{t}{2}}$

Solution
Let's first get a table of values for this function.

t	-2	-1	0	1	2	3
$h(t)$	-35.9453	-21.4084	-12.5914	-7.2436	-4	-2.0327

Here is the sketch.



The main point behind this problem is to make sure you can do this type of evaluation so make sure that you can get the values that we graphed in this example. You will be asked to do this kind of evaluation on occasion in this class.

You will be seeing exponential functions in pretty much every chapter in this class so make sure that you are comfortable with them.

Review : Logarithm Functions

In this section we'll take a look at a function that is related to the exponential functions we looked at in the last section. We will look at logarithms in this section. Logarithms are one of the functions that students fear the most. The main reason for this seems to be that they simply have never really had to work with them. Once they start working with them, students come to realize that they aren't as bad as they first thought.

We'll start with $b > 0$, $b \neq 1$ just as we did in the last section. Then we have

$$y = \log_b x \qquad \text{is equivalent to} \qquad x = b^y$$

The first is called logarithmic form and the second is called the exponential form. Remembering this equivalence is the key to evaluating logarithms. The number, *b*, is called the base.

Example 1 Without a calculator give the exact value of each of the following logarithms.

(a) $\log_2 16$ [Solution]

(b) $\log_4 16$ [Solution]

(c) $\log_5 625$ [Solution]

(d) $\log_9 \dfrac{1}{531441}$ [Solution]

(e) $\log_{\frac{1}{6}} 36$ [Solution]

(f) $\log_{\frac{3}{2}} \dfrac{27}{8}$ [Solution]

Solution

To quickly evaluate logarithms the easiest thing to do is to convert the logarithm to exponential form. So, let's take a look at the first one.

(a) $\log_2 16$

First, let's convert to exponential form.

$$\log_2 16 = ? \qquad \text{is equivalent to} \qquad 2^? = 16$$

So, we're really asking 2 raised to what gives 16. Since 2 raised to 4 is 16 we get,

$$\log_2 16 = 4 \qquad \text{because} \qquad 2^4 = 16$$

We'll not do the remaining parts in quite this detail, but they were all worked in this way.

[Return to Problems]

(b) $\log_4 16$

$$\log_4 16 = 2 \qquad \text{because} \qquad 4^2 = 16$$

Note the difference between the first and second logarithm! The base is important! It can completely change the answer.

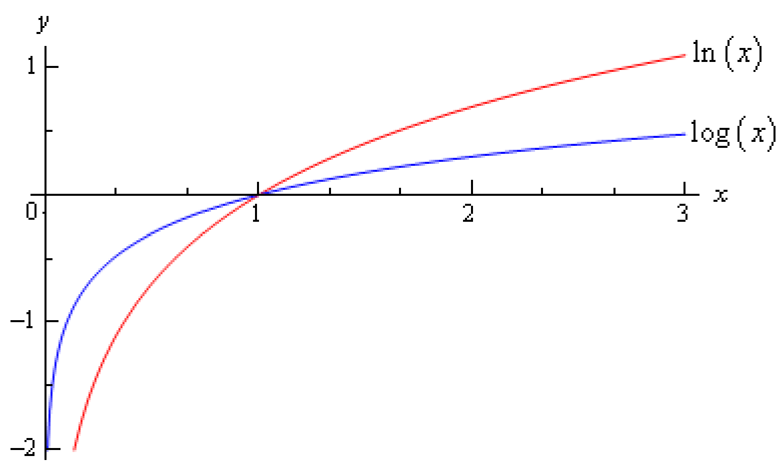
[Return to Problems]

There are a couple of special logarithms that arise in many places. These are,

$\ln x = \log_{\mathbf{e}} x$ This log is called the natural logarithm

$\log x = \log_{10} x$ This log is called the common logarithm

In the natural logarithm the base **e** is the same number as in the natural exponential logarithm that we saw in the last section. Here is a sketch of both of these logarithms.



From this graph we can get a couple of very nice properties about the natural logarithm that we will use many times in this and later Calculus courses.

$$\ln x \to \infty \quad \text{as} \quad x \to \infty$$
$$\ln x \to -\infty \quad \text{as} \quad x \to 0,\ x > 0$$

Let's take a look at a couple of more logarithm evaluations. Some of which deal with the natural or common logarithm and some of which don't.

Example 2 Without a calculator give the exact value of each of the following logarithms.

(a) $\ln \sqrt[3]{\mathbf{e}}$

(b) $\log 1000$

(c) $\log_{16} 16$

(d) $\log_{23} 1$

(e) $\log_{2} \sqrt[7]{32}$

Solution

These work exactly the same as previous example so we won't put in too many details.

(a) $\ln \sqrt[3]{\mathbf{e}} = \dfrac{1}{3}$ because $\mathbf{e}^{\frac{1}{3}} = \sqrt[3]{\mathbf{e}}$

(b) $\log 1000 = 3$ because $10^{3} = 1000$

(c) $\log_{16} 16 = 1$ because $16^{1} = 16$

(d) $\log_{23} 1 = 0$ because $23^{0} = 1$

(e) $\log_{2} \sqrt[7]{32} = \dfrac{5}{7}$ because $\sqrt[7]{32} = 32^{\frac{1}{7}} = \left(2^{5}\right)^{\frac{1}{7}} = 2^{\frac{5}{7}}$

This last set of examples leads us to some of the basic properties of logarithms.

Properties

1. The domain of the logarithm function is $\left(0,\infty\right)$. In other words, we can only plug positive numbers into a logarithm! We can't plug in zero or a negative number.
2. $\log_{b} b = 1$
3. $\log_{b} 1 = 0$
4. $\log_{b} b^{x} = x$
5. $b^{\log_{b} x} = x$

The last two properties will be especially useful in the next section. Notice as well that these last two properties tell us that,

$$f\left(x\right) = b^{x} \qquad \text{and} \qquad g\left(x\right) = \log_{b} x$$

are inverses of each other.

Here are some more properties that are useful in the manipulation of logarithms.

More Properties

6. $\log_b xy = \log_b x + \log_b y$
7. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$
8. $\log_b \left(x^r\right) = r \log_b x$

Note that there is no equivalent property to the first two for sums and differences. In other words,

$$\log_b \left(x + y\right) \neq \log_b x + \log_b y$$

$$\log_b \left(x - y\right) \neq \log_b x - \log_b y$$

The last topic that we need to look at in this section is the **change of base** formula for logarithms. The change of base formula is,

$$\log_b x = \frac{\log_a x}{\log_a b}$$

This is the most general change of base formula and will convert from base *b* to base *a*. However, the usual reason for using the change of base formula is to compute the value of a logarithm that is in a base that you can't easily deal with. Using the change of base formula means that you can write the logarithm in terms of a logarithm that you can deal with. The two most common change of base formulas are

$$\log_b x = \frac{\ln x}{\ln b} \qquad \text{and} \qquad \log_b x = \frac{\log x}{\log b}$$

In fact, often you will see one or the other listed as THE change of base formula!

In the first part of this section we computed the value of a few logarithms, but we could do these without the change of base formula because all the arguments could be written in terms of the base to a power. For instance,

$$\log_7 49 = 2 \qquad \text{because} \qquad 7^2 = 49$$

However, this only works because 49 can be written as a power of 7! We would need the change of base formula to compute $\log_7 50$.

$$\log_7 50 = \frac{\ln 50}{\ln 7} = \frac{3.91202300543}{1.94591014906} = 2.0103821378$$

OR

$$\log_7 50 = \frac{\log 50}{\log 7} = \frac{1.69897000434}{0.845098040014} = 2.0103821378$$

So, it doesn't matter which we use, we will get the same answer regardless of the logarithm that we use in the change of base formula.

Note as well that we could use the change of base formula on $\log_7 49$ if we wanted to as well.

$$\log_7 49 = \frac{\ln 49}{\ln 7} = \frac{3.89182029811}{1.94591014906} = 2$$

This is a lot of work however, and is probably not the best way to deal with this.

So, in this section we saw how logarithms work and took a look at some of the properties of logarithms. We will run into logarithms on occasion so make sure that you can deal with them when we do run into them.

Review : Exponential and Logarithm Equations

In this section we'll take a look at solving equations with exponential functions or logarithms in them.

$$\log_b b^x = x$$

Example 1 Solve $7 + 15\mathbf{e}^{1-3z} = 10$.

Solution

The first step is to get the exponential all by itself on one side of the equation with a coefficient of one.

$$\begin{aligned} 7 + 15\mathbf{e}^{1-3z} &= 10 \\ 15\mathbf{e}^{1-3z} &= 3 \\ \mathbf{e}^{1-3z} &= \frac{1}{5} \end{aligned}$$

Now, we need to get the *z* out of the exponent so we can solve for it. To do this we will use the property above. Since we have an **e** in the equation we'll use the natural logarithm. First we take the logarithm of both sides and then use the property to simplify the equation.

$$\begin{aligned} \ln\left(\mathbf{e}^{1-3z}\right) &= \ln\left(\frac{1}{5}\right) \\ 1 - 3z &= \ln\left(\frac{1}{5}\right) \end{aligned}$$

All we need to do now is solve this equation for *z*.

$$\begin{aligned} 1 - 3z &= \ln\left(\frac{1}{5}\right) \\ -3z &= -1 + \ln\left(\frac{1}{5}\right) \\ z &= -\frac{1}{3}\left(-1 + \ln\left(\frac{1}{5}\right)\right) = 0.8698126372 \end{aligned}$$

Now that we've seen a couple of equations where the variable only appears in the exponent we need to see an example with variables both in the exponent and out of it.

Example 3 Solve $x - x\mathbf{e}^{5x+2} = 0$.

Solution
The first step is to factor an x out of both terms.

DO NOT DIVIDE AN *x* FROM BOTH TERMS!!!!

Note that it is very tempting to "simplify" the equation by dividing an x out of both terms. However, if you do that you'll miss a solution as we'll see.

$$x - x\mathbf{e}^{5x+2} = 0$$
$$x\left(1 - \mathbf{e}^{5x+2}\right) = 0$$

So, it's now a little easier to deal with. From this we can see that we get one of two possibilities.

$$x = 0 \qquad \text{OR}$$
$$1 - \mathbf{e}^{5x+2} = 0$$

The first possibility has nothing more to do, except notice that if we had divided both sides by an x we would have missed this one so be careful. In the second possibility we've got a little more to do. This is an equation similar to the first two that we did in this section.

$$\begin{aligned} \mathbf{e}^{5x+2} &= 1 \\ 5x + 2 &= \ln 1 \\ 5x + 2 &= 0 \\ x &= -\frac{2}{5} \end{aligned}$$

Don't forget that $\ln 1 = 0$!

So, the two solutions are $x = 0$ and $x = -\frac{2}{5}$.

The next equation is a more complicated (looking at least…) example similar to the previous one.

Example 4 Solve $5\left(x^2-4\right)=\left(x^2-4\right)\mathbf{e}^{7-x}$.

Solution

As with the previous problem do NOT divide an x^2-4 out of both sides. Doing this will lose solutions even though it "simplifies" the equation. Note however, that if you can divide a term out then you can also factor it out if the equation is written properly.

So, the first step here is to move everything to one side of the equation and then to factor out the x^2-4.

$$\begin{aligned} 5\left(x^2-4\right)-\left(x^2-4\right)\mathbf{e}^{7-x} &= 0 \\ \left(x^2-4\right)\left(5-\mathbf{e}^{7-x}\right) &= 0 \end{aligned}$$

At this point all we need to do is set each factor equal to zero and solve each.

$$\begin{aligned} x^2-4=0 \qquad\qquad 5-\mathbf{e}^{7-x} &= 0 \\ x=\pm 2 \qquad\qquad\quad \mathbf{e}^{7-x} &= 5 \\ 7-x &= \ln\left(5\right) \\ x &= 7-\ln\left(5\right)=5.390562088 \end{aligned}$$

The three solutions are then $x=\pm 2$ and $x=5.3906$.

Now let's take a look at some equations that involve logarithms. The main property that we'll be using to solve these kinds of equations is,

$$b^{\log_b x} = x$$

Example 6 Solve $3+2\ln\left(\frac{x}{7}+3\right)=-4$.

Solution

This first step in this problem is to get the logarithm by itself on one side of the equation with a coefficient of 1.

$$2\ln\left(\frac{x}{7}+3\right)=-7$$
$$\ln\left(\frac{x}{7}+3\right)=-\frac{7}{2}$$

Now, we need to get the x out of the logarithm and the best way to do that is to "exponentiate" both sides using $\mathbf{e}$. In other word,

$$\mathbf{e}^{\ln\left(\frac{x}{7}+3\right)}=\mathbf{e}^{-\frac{7}{2}}$$

So using the property above with $\mathbf{e}$, since there is a natural logarithm in the equation, we get,

$$\frac{x}{7}+3=\mathbf{e}^{-\frac{7}{2}}$$

Now all that we need to do is solve this for x.

$$\frac{x}{7}+3=\mathbf{e}^{-\frac{7}{2}}$$
$$\frac{x}{7}=-3+\mathbf{e}^{-\frac{7}{2}}$$
$$x=7\left(-3+\mathbf{e}^{-\frac{7}{2}}\right)=-20.78861832$$

At this point we might be tempted to say that we're done and move on. However, we do need to be careful. Recall from the previous section that we can't plug a negative number into a logarithm. This, by itself, doesn't mean that our answer won't work since its negative. What we need to do is plug it into the logarithm and make sure that $\frac{x}{7}+3$ will not be negative. I'll leave it to you to verify that this is in fact positive upon plugging our solution into the logarithm and so $x=-20.78861832$ is in fact a solution to the equation.